The above equation can be interpreted as the HJI equation for a discounted Markovian decision problem. The solution of this discounted Markov decision process may be obtained by either successive approximation or policy improvement techniques; see [5] and [18] for related discussions.

In the next theorem, we show that $v^{\Delta}(x, a, \alpha)$ converges to $v(x, a, \alpha)$ as the step size Δx_j and Δa_i go to zero. For simplicity, we only consider the case that $\Delta x_j = \Delta a_i = \Delta$ for $j = 1, 2, \cdots, P$ and $i = 1, 2, \cdots, M$. A detailed proof can be found in [11].

Theorem 3 (Convergence of the Approximation Scheme): Assume there exists a constant C such that

$$0 \leq v^{\Delta}(x, a, \alpha) \leq C(1 + |x|^{k_g} + |a|^{k_g}).$$

Then

$$\lim_{\Delta \to 0} v^{\Delta}(x, a, \alpha) = v(x, a, \alpha). \quad (6)$$

V. Conclusion

In this paper, we dealt with robust production and maintenance planning of stochastic manufacturing systems. We proved that the value function is the unique viscosity solution of the associated HJI equations, obtained a verification theorem that provides a sufficient condition for optimal control policies, and developed a numerical method for solving the HJI equations. The solution of the numerical procedure was shown to converge to the value function as the step size goes to zero.

The model considered in this paper is relatively simple. It is possible to extend the results to more general models such as flowshops or even jobshops (see [19] and the references therein).

It should be noted that the numerical scheme given in this paper can be used to effectively deal with small or medium sized problems. For large system problems with large dimensional state space, one has to resort to other approximation methods. One of such approximation methods is that of hierarchical decomposition approach. The idea of this approach is to reduce the large complex problem into manageable approximate problems or subproblems, to solve these subproblems, and to construct a solution of the original problem from the solutions of these simpler problems. See [19] for more details on this approach.

References

[1] B. Alziary de Roquefort, "Jeux différentiels et approximation numérique de foncitions valeur, 1re partie: étude théorique," *RAIRO Math. Model Numer. Anal.*, vol. 25, pp. 517–533, 1991.
[2] ——, "Jeux différentiels et approximation numérique de foncitions valeur, 2e partie: étude numérique," *RAIRO Math. Model Numer. Anal.*, vol. 25, pp. 535–560, 1991.
[3] M. Bardi, M. Falcone, and P. Soravia, "Fully discrete schemes for the value function of puesuit-evasion games," in *Advances in Dynamic Games and Applications*, T. Başar and A. Haurie, Eds. Boston, MA: Birkäuser, 1993, pp. 89–105.
[4] T. Başar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems.* Boston, MA: Birkhäuser, 1991.
[5] E. K. Boukas, "Techniques for flow control and preventive maintenance in manufacturing systems," *Contr. Dynamic Syst.*, vol. 48, pp. 327–366, 1991.
[6] ——, "Control of systems with controlled jump Markov disturbances," *Int. J. Contr. Theory Advanced Tech.* vol. 9, pp. 577–595, 1993.
[7] ——, "A stochastic modeling for flexible manufacturing systems," submitted to *IEEE Trans. Robotics Automat.*.
[8] E. K. Boukas and A. Haurie, "Manufacturing flow control and preventive maintenance: A stochastic control approach," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1024–1031, 1990.
[9] E. K. Boukas, Q. Zhu, and Q. Zhang, "A piecewise deterministic Markov process model for flexible manufacturing systems with preventive maintenance," *J. Optim. Theory Appl.*, vol. 81, pp. 259–275, 1994.
[10] ——, "Optimal production and maintenance planning of flexible manufacturing systems," in *Proc. 32nd IEEE Conf. Decis. Contr.*, San Antonio, TX, 1993.
[11] E. K. Boukas, Q. Zhang, and G. Yin, "On robust control problems in stochastic manufacturing systems," to appear in *Recent Advances in Control and Optimization of Manufacturing Systems*, G. Yin and Q. Zhang Eds.
[12] M. G. Crandall, H. Ishii, and P. L. Lion, "User's guide to viscosity solutions of second order partial differential equations," *Amer. Math. Soc. Bull.*, vol. 27, pp. 1–67, 1992.
[13] M. H. A. Davis, *Markov Models and Optimization.* New York: Chapman & Hall, 1993.
[14] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions.* New York: Springer-Verlag, 1992.
[15] J. Q. Hu, P. Vakili, and G. X. Yu, "Optimality of hedging point policies in the production control of failure prone manufacturing systems," *IEEE Trans. Automat. Contr.*, 1994.
[16] J. Q. Hu and D. Xiang, "Monotonicity of optimal flow control for failure prone production systems," to appear in *J. Optim. Theory Appl.*
[17] ——, "Optimal production control for production systems with deterministic up times," preprint, 1993.
[18] H. J. Kushner and P. G. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time.* New York: Springer-Verlag, 1992.
[19] S. P. Sethi and Q. Zhang, *Hierarchical Decision Making in Stochastic Manufacturing Systems.* Boston, MA: Birkhäuser, 1994.
[20] H. M. Soner, "Singular perturbations in manufacturing," *SIAM J. Contr. Optim.*, vol. 31, pp. 132–146, 1993.
[21] Q. Zhang, "Risk sensitive production planning of stochastic manufacturing systems: A singular perturbation approach," *SIAM J. Contr. Optim.*, vol. 33, pp. 498–527, 1995.

W-Stability and Local Input–Output Stability Results

H. Bourlès and F. Colledani

***Abstract*— A new type of input–output stability is defined, based on the use of a Sobolev space W; W is well suited, like the Lebesgue space L_2, to obtain stability characterizations in the time and frequency domains. Moreover, if compared with L_2, W has additional properties which enable us to establish "local" stability results. A local version of the small gain theorem is established in this framework, as well as some consequences of this result, in particular local versions of the passivity theorem and of the circle criterion. The relationship between "W-stability" and asymptotic stability is studied.**

I. Introduction

It is well known that two different types of stability can be considered for a system [17]. The first one is the "internal stability" (e.g., asymptotic stability); the input of the system is then assumed to be zero, and, roughly speaking, one looks if the system state tends to zero from a nonzero initial value; the basic tool to analyze this type

Manuscript received January 3, 1994; revised July 22, 1994.
H. Bourlès is with the Electricité de France, 1, Avenue du Général de Gaulle, 92141 Clamart Cedex, France, and with the Ecole Normale Supérieure de Cachan, 94230 Cachan, France.
F. Colledani is with the Ecole Supérieure d'Electricité, Plateau de Moulon, 91190 Gif sur Yvette, France.
IEEE Log Number 9410789.

0018–9286/95$04.00 © 1995 IEEE

of stability is the use of Lyapunov functions. This approach is well suited to obtain global or local stability results: the internal stability is global when (roughly speaking again) the state tends to zero from any initial value and is local when this occurs for "sufficiently small" initial values of the state.

The second main type of stability is the "input–output stability." In this framework, the initial value of the state is assumed to be zero, so that the system output is a function of the input,[1] and one looks if the output belongs to a specific function space (e.g., a Lebesgue space L_p, $1 \le p \le \infty$) when the input belongs to the same space; moreover, one looks if the norm of the output (in this function space) is smaller than some constant multiplied by the norm of the input. The greatest lower bound of the constants satisfying this property is called the gain of the system. The case $p = 2$ is particularly interesting, because the L_2-gain of any linear time-invariant (LTI) system is the H_∞-norm of its transfer matrix (where H_∞ denotes the well-known Hardy space [7]); moreover, as the Fourier transform is an isomorphism of L_2, this space is well suited to lead to stability characterizations in the time domain as well as in the frequency domain. This approach was limited by the fact that only global stability results are available in the literature; the very aim of this paper is to establish local results in a (more or less) equivalent framework. Note that the internal stability and the input–output one are closely related in the case of reachable and uniformly observable systems [17], [8], [18].

A new input–output approach, leading to global as well as local stability results, has recently been developed; this is the so-called "input-to-state stability" [9], [11], [14], [15]. In this framework (though it seems promising), characterizations of the stability in the frequency domain cannot be obtained, because the function space which is considered is L_∞. Moreover, as L_∞ is not an inner product space, notions such as passivity cannot be defined in this context.

The function space used in this paper is the set of functions x such that x and its derivative belong to L_2. This is the well-known Sobolev space $W_{1,2}$ [16]. This space is interesting for several reasons: on the one hand, it will be proved that all nice properties of L_2 are still satisfied by this space; in particular, the gain of any LTI system remains the H_∞-norm of its transfer matrix. On the other hand, if a function x belongs to $W_{1,2}$ and is smooth enough (more precisely, if it is absolutely continuous [10], [1]), then x belongs to L_∞, and its norm in this latter space is upper bounded by its norm in $W_{1,2}$ if the initial value of x is zero. This is the key property which will enable us to obtain simple conditions for nonlinear input–output operators to be "locally stable" (in a sense precised below).

The paper is organized as follows: in Section II, various signal spaces are defined and their relations are clarified. In Section III, "local W-stability" is defined and then studied in various cases of systems. The notion of "local W-gain" of a system $\mathbf{G}$ is defined and is denoted as $\gamma_{Wl}(\mathbf{G})$. The relationship between W-stability and asymptotic stability is studied in Section IV. In Section V, a local version of the small gain theorem is established. It is proved that if systems $\mathbf{G_1}$ and $\mathbf{G_2}$ in Fig. 1 are locally W-stable and if $\gamma_{Wl}(\mathbf{G_1})\,\gamma_{Wl}(\mathbf{G_2}) < 1$, then the closed-loop system in Fig. 1 is locally W-stable. In Section VI, as consequences of this theorem, local versions of the passivity theorem and of the circle criterion are obtained. A preliminary version has already been published [3].

II. Signal Spaces and Their Relations

The norm of any vector ξ in $\mathbb{R}^n$ or $\mathbb{C}^n$ is denoted as $\|\xi\|$. L_2^n denotes the usual Lebesgue space of functions $x: \mathbb{R}^+ \to \mathbb{R}^n$, which are Lebesgue-measurable and square-integrable; $\langle x, y\rangle_2$ denotes the inner product of two functions x and y in L_2^n, and we set $\|x\|_2 = [\langle x, x\rangle_2]^{1/2}$. For any function $x: \mathbb{R}^+ \to \mathbb{R}^n$, x_T denotes the truncation of x to the interval $[0, T]$ [17], i.e., the function such that $x_T(t) = x(t)$ if $t \le T$, and $x_T(t) = 0$ if $t > T$; L_{2e}^n denotes the "extended L_2^n-space," i.e., the space of functions x such that for any T in $\mathbb{R}^+$, x_T belongs to L_2^n. For any functions x and y belonging to L_{2e}^n, we set $\langle x, y\rangle_{2,T} = \langle x_T, y_T\rangle_2$ and $\|x\|_{2,T} = \|x_T\|_2$. In the same manner, L_∞^n denotes the space of functions $x: \mathbb{R}^+ \to \mathbb{R}^n$, which are Lebesgue-measurable and essentially bounded; the norm in this space is defined as $\|x\|_\infty = \text{ess.sup}\{\|x(t)\|; t \in \mathbb{R}^+\}$. The "extended L_∞^n-space," $L_{\infty e}^n$, is the space of functions x such that for any T in $\mathbb{R}^+$, x_T belongs to L_∞^n, and $\|x\|_{\infty,T}$ denotes the quantity $\|x_T\|_\infty$.

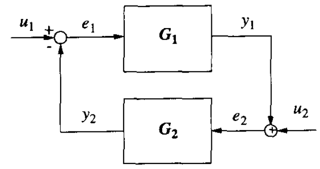

Fig. 1. Standard closed-loop system.

$W_{1,2}^n$ denotes the Sobolev space of functions $x: \mathbb{R}^+ \to \mathbb{R}^n$ such that x and its distributional-derivative $\dot{x}$ belong to L_2^n. The inner product of two functions x and y in $W_{1,2}^n$ is defined as $\langle x, y\rangle_W = \langle x, y\rangle_2 + \langle \dot{x}, \dot{y}\rangle_2$ and the norm of x in $W_{1,2}^n$ is defined by $\|x\|_W = [\langle x, x\rangle_W]^{1/2}$. Obviously, $W_{1,2}^n$ is included in L_2^n, and for any function x in $W_{1,2}^n$ one has $\|x\|_2 \le \|x\|_W$. Now, AC^n denotes the set of functions $x: \mathbb{R}^+ \to \mathbb{R}^n$ which are absolutely continuous. By Lebesgue's theorem [10], if x belongs to AC^n, then x is almost everywhere differentiable (in the usual sense), and its derivative $\dot{x}$ is integrable on any bounded interval $[a, b]$ included in $\mathbb{R}^+$; moreover, one has $\int_a^b \dot{x}(\tau)d\tau = x(b) - x(a)$. The following facts are well known [1]: AC^n is a vector space; if x and y belong to AC^n, then the function $t \to x(t)^T y(t)$ belongs to AC^1. Let be a function $\Phi: \Omega \to \mathbb{R}^m$, where Ω denotes some open subset of $\mathbb{R}^n$; if Φ is Lipschitz-continuous and if x belongs to AC^n and takes its values in Ω, then the function $t \to \Phi(x(t))$ belongs to AC^m.

Definition 1: W^n is the space of functions x belonging to $W_{1,2}^n \cap \mathrm{AC}^n$ and such that $x(0) = 0$; W_e^n is the space of functions x belonging to AC^n, and such that $x(0) = 0$ and for any finite $T > 0$, x_T and $(\dot{x})_T$ belong to L_2^n. For any functions x and y belonging to W_e^n, we set $\langle x, y\rangle_{W,T} = \langle x, y\rangle_{2,T} + \langle \dot{x}, \dot{y}\rangle_{2,T}$ and $\|x\|_{W,T} = [\langle x, x\rangle_{W,T}]^{1/2}$.

Note that x_T is not absolutely continuous, so that $\|x_T\|_W$ cannot be defined. Obviously, $\|\cdot\|_W$ is a norm on W^n, whereas $(\|\cdot\|_{W,T})_{T\ge 0}$ is a family of seminorms on W_e^n. (These spaces can be replaced by their completions and then become, respectively, a Hilbert space and a Fréchet space [16].)

Proposition 1: Let $x \in W_e^n$; then for any finite $T > 0$ the following inequality holds

$$\|x\|_{\infty,T} \le \|x\|_{W,T}. \tag{1}$$

W^n is included in L_∞^n; for any $x \in W^n$, one has $\|x\|_\infty \le \|x\|_W$ and $x(t)$ tends to zero as t tends to ∞.

Proof: The function $\tau \to \|x(\tau)\|^2$ belongs to AC^1, hence one has for any finite $t > 0$: $2\langle x, \dot{x}\rangle_t = \|x(t)\|^2 - \|x(0)\|^2 = \|x(t)\|^2$. Let $T \ge t$: by Schwarz's inequality, $\|x(t)\|^2 \le 2\|x\|_{2,t}\|\dot{x}\|_{2,t} \le (\|x\|_{W,t})^2 \le (\|x\|_{W,T})^2$. Moreover, if $x \in W^n$, then $x(t)$ tends to zero as t tends to ∞ ([19, Lemma 3.2]), hence $x \in L_\infty^n$. Therefore, the proposition is proved by taking T tending to ∞ in (1). □

[1] In what follows, this function is called the input–output operator associated to the system.

For every $u \in L_{2e}^n$, set $\mathbf{K_1} u = y$, with $y(t) = \int_0^t e^{-(t-\tau)} u(\tau) d\tau$ (so that $\mathbf{K_1}$ is the causal LTI operator with transfer matrix $K_1(s) = (1+s)^{-1} I_n$, where I_n denotes the identity matrix of dimension n). The following result gives a useful characterization of signals in W^n.

Proposition 2: $\mathbf{K_1}$ is one-to-one from L_{2e}^n onto W_e^n and $\mathbf{K}_1^{-1}$ is the operator $y \to y + \dot{y}$;[2] for any finite $T > 0, \|y\|_{W,T} \leq \|u\|_{2,T}$; $y = \mathbf{K_1} u$ belongs to W_n if and only if (iff) $u \in L_2^n$, and then $\|y\|_W = \|u\|_2$.

Proof: Let $y = \mathbf{K_1} u$, where $u \in L_{2e}^n$ (respectively, L_2^n). First, note that $y \in L_{2e}^n$ (respectively, L_2^n) [16, Chap. 26]. Obviously, $y(0) = 0, y \in AC^n$, and $y + \dot{y} = u$, hence $\dot{y} \in L_{2e}^n$ (respectively, L_2^n). Therefore, $y \in W_e^n$ (respectively, W^n) and $\mathbf{K}_1^{-1}$ is given by the formula above. Note that $\mathbf{K_1}^{-1} y \in L_{2e}^n$ (respectively, L_2^n) for every $y \in W_e^n$ (respectively, W^n). For any $u \in L_{2e}^n$, it is easy to verify that $(\|u\|_{2,T})^2 = (\|\dot{y}\|_{2,T})^2 + (\|y\|_{2,T})^2 + \|y(T)\|^2$, hence $\|y\|_{W,T} \leq \|u\|_{2,T}$. If $u \in L_2^n$, then $y(T) \to 0$ as $T \to \infty$ because $y \in W^n$, thus $\|y\|_W = \|u\|_2$. □

The notion of W-gain can be defined as usually [6]: let be $\mathbf{G}: W_e^n \to W_e^m$, and $K = \{k > 0 : \|\mathbf{G}u\|_{W,T} \leq k\|u\|_{W,T}, \forall u \in W_e^n, \forall T > 0\}$. If K is nonempty, then $\mathbf{G}$ is said to be W-stable (W-s), and $\gamma_W(\mathbf{G}) = \inf(K)$ is called the W-gain of $\mathbf{G}$ (note that, according to the terminology used in [6], this is the definition of the W-gain with zero bias); if K is empty, we set $\gamma_W(\mathbf{G}) = \infty$.

All operators considered in this paper are assumed to be causal, hence one has [19]

$$\gamma_W(\mathbf{G}) = \sup_{u \in W^n - \{0\}} \frac{\|\mathbf{G}u\|_W}{\|u\|_W}. \tag{2}$$

Let be $\mathbf{G}: W_e^n \to W_e^m$; by Proposition 2, one has the following result, where γ_2 denotes the L_2-gain.

Proposition 3: $\mathbf{G}$ is W-stable iff $\mathbf{K}_1^{-1}\mathbf{G}\mathbf{K_1}$ is L_2-stable, and $\gamma_W(\mathbf{G}) = \gamma_2(\mathbf{K}_1^{-1}\mathbf{G}\mathbf{K_1})$.

Remark 1: In particular, assume that $\mathbf{G}$ is a LTI operator with transfer matrix G; as $\mathbf{G}$ and $\mathbf{K_1}$ commute, $\mathbf{G}$ is W-stable iff it is L_2-stable, and one has $\gamma_W(\mathbf{G}) = \gamma_2(\mathbf{G}) = \|G\|_\infty$ (note that with respect to transfer matrices, $\|\cdot\|_\infty$ denotes the norm in H_∞, whereas with respect to time domain signals, this symbol denotes the norm in L_∞).

For any operator $\mathbf{G}: L_{2e}^n \to W_e^m$, one has by Proposition 2

$$\sup_{u \in L_2^n - \{0\}} \frac{\|\mathbf{G}u\|_W}{\|u\|_2} = \gamma_2(\mathbf{K_1}^{-1}\mathbf{G}) \tag{3}$$

(this quantity being finite iff $\mathbf{K}_1^{-1}\mathbf{G}$ is L_2-stable). In particular, for $\tau > 0$, let $\mathbf{K}_\tau$ be the LTI operator with transfer matrix $K_\tau(s) = (1+\tau s)^{-1} I_n$. Obviously, $\mathbf{K}_1^{-1}\mathbf{K}_\tau$ is L_2-stable, so that for any function $u \in L_2^n$, $\mathbf{K}_\tau u \in W^n$. Hence, W^n (respectively, W_e^n) is the space of functions in L_2^n (respectively, L_{2e}^n) filtered by first order low-pass filters. By Proposition 3, one has $\min(\tau, \tau^{-1})\gamma_2(\mathbf{K}_\tau^{-1}\mathbf{G}\mathbf{K}_\tau) \leq \gamma_W(\mathbf{G}) \leq \max(\tau, \tau^{-1})\gamma_2(\mathbf{K}_\tau^{-1}\mathbf{G}\mathbf{K}_\tau)$. The following result is a consequence of (3) and Proposition 1 and can be of practical interest.

Corollary 1: Assume that $\mathbf{G}: L_{2e}^n \to W_e^m$ is such that $\mathbf{K_1}^{-1}\mathbf{G}$ is L_2-stable, and set $y = \mathbf{G}u$: then, y belongs to L_∞^m if u belongs to L_2^n, and $\|y\|_\infty \leq \gamma_2(\mathbf{K_1}^{-1}\mathbf{G})\|u\|_2$.

III. Local W-Stability

Definition 2: Let be $\mathbf{G}: W_e^n \to W_e^m$ and $K_l = \{k > 0, \exists \varepsilon > 0 : \|\mathbf{G}u\|_{W,T} \leq k\|u\|_{W,T}$ whenever $u \in W_e^n$ and $T > 0$ are such that $\|u\|_{W,T} < \varepsilon\}$. If K_l is nonempty, $\mathbf{G}$ is said to be locally-W-stable (l-W-s) and $\gamma_{Wl}(\mathbf{G}) = \inf(K_l)$ is called the local-W-gain (l-W-g) of $\mathbf{G}$. If K_l is empty, we set $\gamma_{Wl}(\mathbf{G}) = \infty$.[3]

Remark 2: This definition of the "local gain" is close, but not identical to the definition of the "small-signal-gain" in [17]. The latter definition corresponds to the case where the inequality $\|u\|_{W,T} < \varepsilon$ above is replaced by $\|u\|_{\infty,T} < \varepsilon$. By Proposition 1, any small-signal-W-stable system is l-W-s. Obviously, if $\mathbf{G}$ is linear, then $\mathbf{G}$ is l-W-s iff $\mathbf{G}$ is W-s, and $\gamma_{Wl}(\mathbf{G}) = \gamma_W(\mathbf{G})$.

First, let us consider the case of a nonlinear memoryless operator $\mathbf{G}$, defined as

$$(\mathbf{G}u)(t) = \Phi(t, u(t)) \quad \forall t \geq 0 \tag{4}$$

where $\Phi: \mathbb{R}^+ \times \Omega \to \mathbb{R}^m$ (Ω denoting some open neighborhood of zero in $\mathbb{R}^n$) is a function satisfying $\Phi(t, 0) = 0, \forall t$, and where u is any function in W_e^n, taking its values in Ω. Let ∂_t and ∂_u, respectively, denote the partial derivative with respect to t and u. The following notion will be useful.

Definition 3: Let $u^* \in \Omega$; Φ is said to be differentiable with respect to u at point u^* uniformly with respect to t, if there exists a function $\eta: \Omega \to \mathbb{R}^+$ such that $\eta(u^* + h)$ tends to zero as h tends zero and that $\forall t \geq 0$

$$\|\Phi(t, u^* + h) - \Phi(t, u^*) - \partial_u \Phi(t, u^*) h\| = \eta(u^* + h)\|h\|. \tag{5}$$

Lemma 1: Assume that Φ is differentiable with respect to u at point $u^* = 0$ uniformly with respect to t and is locally Lipschitz-continuous at point zero, with Lipschitz constant k_Φ: obviously

$$\sup_{t \geq 0} \bar{\sigma}(\partial_u \Phi(t, 0)) \leq k_\Phi \tag{6}$$

(where $\bar{\sigma}$ denotes the maximum singular value).

Proposition 4: Assume that Φ is C^1 and satisfies the assumptions of Lemma 1 and the following condition

$$\lim \cdot \sup_{u \to 0} \frac{\sup_{t \geq 0} \|\partial_t \Phi(t, u)\|}{\|u\|} = \beta < \infty. \tag{7}$$

Then, $\mathbf{G}$ is l-W-s and

$$\gamma_{Wl}(\mathbf{G}) \leq \beta + \sup_{t \geq 0} \bar{\sigma}(\partial_u \Phi(t, 0)). \tag{8}$$

Proof: Let $y = \mathbf{G}u$, so that for all $t \geq 0, y(t) = \Phi(t, u(t))$ and $\dot{y}(t) = \partial_t \Phi(t, u(t)) + \partial_u \Phi(t, u(t)) \dot{u}(t)$. By Definition 3, one has $\lim \cdot \sup_{u \to 0} \frac{\sup_{t \geq 0} \|\Phi(t, u)\|}{\|u\|} \leq \alpha$, where $\alpha = \sup_{t \geq 0} \bar{\sigma}(\partial_u \Phi(t, 0))$ (note that α is finite by Lemma 1). Hence, for any $\delta > 0$ and any $T > 0$, there exists $\varepsilon > 0$ such that for all $t \in [0, T], \|u(t)\| < \varepsilon \Rightarrow \|y(t)\| \leq \alpha'\|u(t)\|$ and $\|\dot{y}(t)\| \leq \beta'\|u(t)\| + \alpha'\|\dot{u}(t)\|$, where $\alpha' = \alpha + \delta$ and $b' = \beta + \delta$. Therefore, $\|y(t)\|^2 + \|\dot{y}(t)\|^2 \leq (\alpha' + \beta')^2(\|u(t)\|^2 + \|\dot{u}(t)\|^2)$. Hence, the result is proved by using Proposition 1 and by taking δ tending to zero. □

Example 1: In the case $n = m = 1$, assume that $\Phi(t, u) = \cos \omega t (e^u - 1), \omega \geq 0$; then $\gamma_{Wl}(\mathbf{G}) \leq \omega + 1$.

Now, let us consider the case of an operator $\mathbf{G}$ associated to a nonlinear time-invariant system Σ described by state-space equations $\dot{x} = f(x, u), y = g(x, u)$ (see footnote 1), where f and g are C^1 in

[2] As $y \in W_e^n$, the derivative $\dot{y}$ is defined almost everywhere in the usual sense. Note that $\mathbf{K_1}^{-1}$ is causal.

[3] Note that $\gamma_{Wl}(\mathbf{G} = \inf_{\varepsilon > 0} \sup_{0 < \|u\|_W < \varepsilon} \frac{\|\mathbf{G}u\|_W}{\|u\|_W}$, because $\mathbf{G}$ is causal. Obviously, γ_{Wl} is a seminorm, whereas γ_W is a norm; both of them are submultiplicative.

a neighborhood of $(0,0)$ and satisfy $f(0,0) = 0,\ g(0,0) = 0$. Set $\partial_x f(0,0) = A,\ \partial_u f(0,0) = B, \partial_x g(0,0) = C,\ \partial_u g(0,0) = D$, and let $G(s)$ be the transfer matrix of the linear approximation Σ_l of Σ around $(0,0)$.

Theorem 1: Assume that (C, A) is detectable, (A, B) is stabilizable and $G \in H_\infty$. Then, Σ is l-W-s and $\gamma_{Wl}(\Sigma) = \|G\|_\infty$.[4]

Proof: As $G \in H_\infty$, Σ_l is W-s by Remark 1. Hence, zero is an exponentially stable equilibrium for Σ_l (because $(C,\ A)$ is detectable and $(A,\ B)$ is stabilizable). As a result, by Lyapunov's theorem, zero is exponentially stable for Σ and for the system Σ_0 with input u and output x and defined by $\dot{x} = f(x, u)$. Therefore, there exist $\varepsilon > 0$ and $\gamma > 0$ such that $\|u\|_W < \varepsilon \Rightarrow \|x\|_W \leq \gamma\|u\|_W$. The equation of Σ_0 can be written $\dot{x} = Ax + Bu + v$, where $v = f(x, u) - Ax - Bu$. Therefore, $x = x_1 + x_2$, where x_1 and x_2 are the outputs of the systems Σ_1 and Σ_2, respectively, defined by $\dot{x}_1 = Ax_1 + Bu$ and $\dot{x}_2 = Ax_2 + Bv$. Now, consider the operator $\mathbf{P}$ defined by $\mathbf{P}(x, u) = f(x, u) - Ax - Bu$, and set $z = (x, u)$, so that $v = \mathbf{P}z$. By Proposition 4, $\gamma_{Wl}(\mathbf{P}) = 0$. Hence, for every $\eta > 0$, there exists $\delta > 0$ such that $\|z\|_W < \delta \Rightarrow \|v\|_W \leq \eta\|z\|_W \leq \eta(\|x\|_W + \|u\|_W)$. Take $\|u\|_W < \varepsilon$; then $\|v\|_W \leq \eta(1 + \gamma)\|u\|_W$, hence for every $\gamma' > \gamma_{Wl}(\Sigma_1)$ one has $\|x_2\|_W \leq \eta(1+\gamma)\gamma'\|u\|_W$ and $\|x\|_W \leq [1+\eta(1+\gamma)]\gamma'\|u\|_W$. As η can be taken arbitrarily small, this proves that $\gamma_{Wl}(\Sigma_0) \leq \gamma_{Wl}(\Sigma_1)$. By a similar rationale, one obtains $\gamma_{Wl}(\Sigma_0) = \gamma_{Wl}(\Sigma_1)$. Now, consider the system Σ_3 with input u and output y, and defined by $\dot{x} = f(x, u), y = Cx + Du$. By Proposition 4, $\gamma_{Wl}(\Sigma_3) = \gamma_{Wl}(\Sigma)$, and the equality $\gamma_{Wl}(\Sigma_0) = \gamma_{Wl}(\Sigma_1)$ implies $\gamma_{Wl}(\Sigma_3) = \gamma_{Wl}(\Sigma_l)$. □

IV. Relationship between W-Stability and Asymptotic Stability

A. Time-Varying Case

Proposition 5: Consider the system Σ, described by the state-space equations

$$\dot{x} = f(t, x, u) \tag{9}$$

$$y = g(t, x, u) \tag{10}$$

$x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in R^p$, where f and g are C^1 and are locally Lipschitz-continuous at $z^* = 0$, where $z = [x^T u^T]^T$, and $f(t, 0, 0) = 0$ and $g(t, 0, 0) = 0,\ \forall t$. In addition, assume that g is differentiable with respect to z at point zero, uniformly with respect to t, and that

$$\lim_{z \to 0} \cdot \sup \frac{\sup_{t \geq 0} \|\partial_t g(t, z)\|}{\|z\|} < \infty. \tag{11}$$

Then, if zero is a locally exponentially stable equilibrium point of (9) (with $u = 0$), the operator $\mathbf{G}$ associated to Σ is l-W-s.

Proof: By Theorem 15 in [17, Section 6.3], the operator $u \to x$ is small-signal-L_2-stable, hence there exist $k > 0$ and $\varepsilon > 0$ such that $\forall u \in W_e^n,\ \forall T > 0,\ \|u\|_{\infty,T} < \varepsilon \Rightarrow \|x\|_{2,T} \leq k\|u\|_{2,T}$. This implies that $\|x\|_{2,T} \leq k\|u\|_{W,T}$. Let k_f and k_g, respectively, denote the Lipschitz constants of f and g. If ε is small enough, then for any $t \in [0,\ T]$, one has $\|\dot{x}(t)\| \leq k_f(\|x(t)\| + \|u(t)\|)$ (see the proof of the cited theorem); therefore, $\|\dot{x}(t)\|^2 \leq 2k_f^2(\|x(t)\|^2 + \|u(t)\|^2)$. Hence, $(\|\dot{x}\|_{2,T})^2 \leq 2k_f^2[(\|x\|_{2,T})^2 + (\|u\|_{2,T})^2] \leq 2k_f^2(k^2 + 1)(\|u\|_{2,T})^2$. Finally, $(\|x\|_{2,T})^2 + (\|\dot{x}\|_{2,T})^2 \leq [k^2 + 2k_f^2(k^2 + 1)](\|u\|_{2,T})^2$, so that $\|x\|_{W,T} \leq [k^2 + 2k_f^2(k^2+1)]^{1/2}\|u\|_{W,T}$. As a result, by Proposition 1, the operator $u \to x$ is l-W-s. Proposition 5 is now a consequence of Proposition 4. □

[4]For the sake of simplicity, the system Σ and the associated operator are denoted by the same symbol.

This result is very similar to the theorem cited above; the main difference (apart from the fact that W-stability, instead of L_p-stability is obtained) is condition (11). For instance, in the case of a linear output equation $y = C(t)x$, this condition reduces to $\sup_{t \geq 0} \bar{\sigma}(\dot{C}(t)) < \infty$.

B. Time-Invariant Case

We now assume that Σ is time-invariant, i.e., $f(t, x, u) = f(x, u)$, $g(t, x, u) = g(x, u)$, where f and g are C^1 in a neighborhood of $(0,0)$ and satisfy $f(0,0) = 0,\ g(0,0) = 0$. Let us denote as $\Phi\ (t, t_0, x_0, u)$ the state $x(t)$ satisfying the initial condition $x(t_0) = x_0$.

Definition 4: Σ is locally W-reachable if there exist a function β of class K [17] and a neighborhood U of zero in $\mathbb{R}^n$ such that for any x in U, there exists a finite time τ and a control $u \in W^m$ such that $\Phi(\tau, 0, 0, u) = x, u(t) = 0$ for $t \geq \tau$ and $\|u\|_W \geq \beta(\|x\|)$. If $U = \mathbb{R}^n, \Sigma$ is said to be globally W-reachable.

W-reachability is a natural generalization of controllability in the case of linear systems possibly fed back by a time-invariant memoryless nonlinearity (see Proposition 6 and Remark 3 below).

Proposition 6: Assume that Σ is linear, i.e. $f(x, u) = Ax + Bu$, $g(x, u) = Cx + Du$. Then, (A, B) is controllable iff Σ is globally W-reachable.

Proof: 1) Obviously, if (A, B) is not controllable, then Σ is not W-reachable. 2) Conversely, assume that (A, B) is controllable. Set $\dot{u} = v$ and $X = [x^T u^T]^T; X$ and v obey to the state-space equation $\dot{X} = FX + Gv$, where the meanings of F and G are obvious. Clearly, (F, G) is controllable. Let $\psi(t, t_0, X_0, v)$ be the state $X(t)$ satisfying the initial condition $X(t_0) = X_0$, and let $\mathcal{V}$ be the linear space of functions $v: \mathbb{R}^+ \to \mathbb{R}^m$ which are continuous and satisfy $v(t) = 0$ for $t \geq 1$. The map $L: \mathcal{V} \to \mathbb{R}^{n+m}: v \to \psi(1, 0, 0, v)$ is linear and epic; therefore, it possesses a right inverse M. Now, let $W^m([0,1])$ denote the linear space of absolutely continuous functions $u: [0,1] \to \mathbb{R}^m$ such that $u(0) = 0$ and u and $\dot{u}$ are square-integrable on $[0,1]$; the norm of a function u in $W^m([0,1])$ is the quantity $\|u\|_{W,1}$ defined above. Let $N: \mathcal{V} \to W^m([0,1])$ be the linear map defined by $Nv = u$ with $u(t) = \int_0^t v(\sigma)\, d\sigma$ for $t \leq 1$; NoM is linear and continuous on $\mathbb{R}^{n+m}$ to the normed space $W^m([0,1])$,[5] hence it has a finite induced norm $\|NoM\|$. Finally, let x be any vector in $\mathbb{R}^n$, set $X = (x, 0)$, and define u such that $u(t) = (NoM)(X)(t)$ if $t < 1, u(t) = 0$ if $t \geq 1$; then $u \in W^m, \phi(1, 0, 0, u) = x, \|u\|_W = \|u\|_{W,1} \leq \|NoM\|\|X\|$ and $\|X\| = \|x\|$. Hence, Σ is globally W-reachable. □

Remark 3: Consider the feedback system in Fig. 1, where $\mathbf{G_1}$ is the system Σ of Proposition 6 and where $\mathbf{G_2}$ is a time-invariant memoryless nonlinearity defined by $\mathbf{G_2}(u)(t) = \Phi(u(t))$, where Φ is Lipschitz-continuous in a neighborhood U of zero in $\mathbb{R}^m$ and satisfies $\Phi(0) = 0$. Assume that $u_2 = 0$ and consider the system Σ_f with input u_1. By a rationale similar to the one above (see also [17, Section 6.3, Theorem 46]), it is easy to prove that Σ_f is W-reachable (locally is U is strictly included in $\mathbb{R}^m$, and globally if $U = \mathbb{R}^m$) if (A, B) is controllable.

Recall that the nonlinear time-invariant system Σ is said to be locally uniformly observable iff there exist a function α of class K and a neighborhood V of zero in $\mathbb{R}^n$ such that $\forall x \in V, \|g(\phi(\cdot, 0, x, 0), 0)\|_2 \geq \alpha(\|x\|)$; if $V = \mathbb{R}^n$, then Σ is said to be globally uniformly observable [18]. For LTI systems, uniform observability is equivalent to the standard notion of observability; moreover, the closed-loop system Σ_f above is globally uniformly observable if (C, A) is observable ([17, Section 6.3, Theorem 46]).

[5]Recall that every linear map on a finite-dimensional vector space to a possibly infinite-dimensional topological vector space is continuous.

The theorem below can be considered as the reciprocal of Proposition 5 (with additional minimality assumptions on the state-space realization).

Theorem 2: Assume that the nonlinear time-invariant system Σ is locally (respectively, globally) W-reachable, locally (respectively, globally) uniformly observable and l-W-s (respectively, W-s). Then, zero is a locally (respectively, globally) asymptotically stable equilibrium for the unforced system.

Proof: By hypothesis (in the local case) there exist $\varepsilon > 0$ and $\gamma > 0$ such that $\|u\|_W < \varepsilon \Rightarrow \|y\|_W \leq \gamma\|u\|_W$. Let $\delta > 0$ and $\varepsilon' > 0$ be such that $\beta(\delta) < \varepsilon' \leq \varepsilon$, and let x_0 be a state such that $\|x_0\| < \min(\delta, r)$, where $r > 0$ is such that the ball $\|x\| < r$ is included in $U \cap V$ (where U and V are the neighborhoods used in the definitions of local W-reachability and local uniform observability). Then, there exist a time $\tau > 0$ and a control u satisfying the conditions of Definition 4, i.e., $\|u\|_W \leq \beta(\|x_0\|) < \varepsilon'$ and $\phi(\tau, 0, 0, u) = x_0$. Therefore, $y \in W^p$ and $\|y\|_W \leq \gamma\|u\|_W \leq \gamma\varepsilon'$. As Σ is time-invariant, the time $t = \tau$ can be considered as the initial time (note that $x(\tau) = x_0$). Now, as Σ is locally uniformly observable, there exists a function α of class K such that $\|y - y_t\|_2 \geq \alpha(\|x(t)\|)$ while $\|x(t)\| < r$. For these values of $t \geq \tau$, one has $\|y\|_W \geq \|y\|_2 \geq \|y - y_t\|_2 \geq \alpha(\|x(t)\|)$, hence $\|x(t)\| \leq \alpha^{-1}(\gamma\varepsilon')$. Assume that ε' has been chosen sufficiently small for having $\alpha^{-1}(\gamma\varepsilon') < r$; then, by contradiction it follows that the inequalities above hold for all values of $t \geq \tau$ (because $x(\cdot)$ is continuous). Therefore, zero is Lyapunov-stable. Now, $W^p \subset L_2^p$, hence $\|y - y_t\|_2 \to 0$ as $t \to \infty$. As a result, zero is attractive, because $\alpha(\|x(t)\|) \to 0$ implies $x(t) \to 0$. This proof still holds in the global case with the appropriate modifications. □

V. Local Small Gain Theorem

Let us consider the standard closed-loop system in Fig. 1, where $\mathbf{G_1}: W_e^n \to W_e^m$ and $\mathbf{G_2}: W_e^m \to W_e^n$ are l-W-s; set $u = [u_1^T \; u_2^T]^T$, $y = [y_1^T \; y_2^T]^T$ and $e = [e_1^T \; e_2^T]^T$. All signals are assumed to be zero at initial time zero. Assume that this closed-loop system is well posed [19], i.e., there exist two operators $\mathbf{H_1}$ and $\mathbf{H_2}: W_e^{n+m} \to W_e^{n+m}$ such that $e = \mathbf{H_1}u$ and $y = \mathbf{H_2}u$. Let us say that this closed-loop system is l-W-s iff $\mathbf{H_1}$ and $\mathbf{H_2}$ are l-W-s.

The following theorem is a local version of the well known Small Gain Theorem [20].

Theorem 3: If $\gamma_{Wl}(\mathbf{G_1})\gamma_{Wl}(\mathbf{G_2}) < 1$. then the closed-loop system is l-W-s.

Proof: By hypothesis, there exist $\varepsilon_i > 0$ and $\gamma_i > 0 (i = 1, 2)$ such that $\|e_i\|_{W,T} < \varepsilon_i \Rightarrow \|y_i\|_{W,T} \leq \gamma_i\|e_i\|_{W,T}$, with $\gamma_1 \gamma_2 < 1$. Let $\varepsilon = \min\{\frac{\varepsilon_i}{\lambda_i\sqrt{2}}, i = 1, 2\}$, where $\lambda_i = \max\{\frac{1}{1-\gamma_1\gamma_2}, \frac{\gamma_j}{1-\gamma_1\gamma_2}\}$, $i, j = 1, 2$, $i \neq j$. For any $\eta > 0$, let $\mathcal{T}_\eta = \{T > 0, \exists u \in W_e^{n+m} : \|u\|_{W,T} < \eta$ and $\|e_1\|_{W,T} \geq \varepsilon_1$ or $\|e_2\|_{W,T} \geq \varepsilon_2\}$. Take for η any positive real such that $\eta < \varepsilon$. 1) If $\mathcal{T}_\eta$ is empty, then $\forall T > 0, \forall u \in W_e^{n+m}: \|u|_{W,T} < \eta \Rightarrow \|e_1\|_{W,T} < \varepsilon_1$ and $\|e_2\|_{W,T} < \varepsilon_2$. Therefore, $\forall T, \|y_i\|_{W,T} \leq \gamma_i\|e_i\|_{W,T} (i = 1, 2)$. One has $e_1 = u_1 - y_2$ and $e_2 = u_2 + y_1$, hence $\|e_i\|_{W,T} \leq \|u_i\|_{W,T} + \gamma_j\|e_j\|_{W,T}$, where $i, j = 1$ or 2 and $i \neq j$. Therefore, $\|e_i\|_{W,T} \leq (1-\gamma_1\gamma_2)^{-1}[\|u_i\|_{W,T} + \gamma_j\|u_j\|_{W,T}] \leq \lambda_i\sqrt{2}\|u\|_{W,T} < \varepsilon_i$. Hence, $\|y_i\|_{W,T} \leq \gamma_i\lambda_i\sqrt{2}\|u\|_{W,T}$, so that $\|y\|_{W,T} \leq \delta\|u\|_{W,T}$, where $\delta = \max(\gamma_1\lambda_1, \gamma_2\lambda_2)$. 2) Now, let us prove by contradiction that $\mathcal{T}_\eta$ is empty. If not, let $t_\eta = \inf(\mathcal{T}_\eta)$. First, note that $t_\eta > 0$, because $e(0) = 0$ and e is continuous (by definition of W_e^{n+m}). For $0 \leq T < t_\eta$, the inequalities above hold; hence, $\|e_i\|_{W,T} < \lambda_i\sqrt{2}\eta < \varepsilon_i$. For $T \geq t_\eta$, there exists u such that $\|u\|_{W,T} < \eta$ and $\|e_1\|_{W,T} \geq \varepsilon_1$ or $\|e_2\|_{W,T} \geq \varepsilon_2$; hence, one at least of the functions $T \to \|e_i\|_{W,T} (i = 1$ or $2)$ is discontinuous at time t_η, but this is impossible. □

Remark 4: This theorem is still true (with a similar proof) if L_p-stability $(1 \leq p < \infty)$ is considered instead of W-stability. W-stability is more interesting, however, because Proposition 4 and Theorem 1 do not hold in the L_p framework. Consider, for instance, the case where one of the operators in the loop is memoryless and only locally bounded (in the sense where the associated function Φ satisfies the assumptions of Proposition 4); a signal x in L_2 can take very large values, even if $\|x\|_2$ is very small, and then nothing can be said about the signal $t \to \Phi(t, x(t))$; hence, local L_2-stability cannot be established (in an input–output approach). Small-signal-L_2-stability can be proved in this case by first proving internal stability (via Lyapunov theory) and then using the theorem cited in the proof of Proposition 5. But to prove internal stability can be difficult, especially when one of the systems in the loop is infinite dimensional (see all examples below).

Example 1 (Continued): Assume that $\mathbf{G_1}$ is the LTI operator with transfer function $k\frac{e^{-\tau s}}{1+T_s}(\tau > 0, T > 0)$ and that $\mathbf{G_2}$ is an operator of the form (4), where $\Phi(t, u) = \cos \omega t(e^u - 1)$, $\omega \geq 0$. By Theorem 3 and Proposition 4, the resulting closed-loop system is l-W-s if $|k| < \frac{1}{1+\omega}$.

Theorem 3 gives a sufficient condition for l-W-stability; conversely, a necessary condition can also be obtained.

Corollary 2: Let $\mathbf{G_1}: W_e^n \to W_e^m$ be a LTI operator. The closed-loop system in Fig. 1 is l-W-stable for any operator $\mathbf{G_2}: W_e^m \to W_e^n$ satisfying $\gamma_{Wl}(\mathbf{G_2}) < 1$, iff $\gamma_W(\mathbf{G_1}) \leq 1$.

Proof: By Remark 1, if the latter condition is not satisfied, it is possible to find a LTI operator $\mathbf{G_2}$ such that the closed-loop system is unstable [4]. □

VI. Local Passivity and Local Sector Conditions

A. Local Passivity

As W^n is an inner product space, the notions of W-passivity and strict W-passivity can obviously be defined [19], [6]. Local versions of these notions will now be defined. Let be $\mathbf{G}: W_e^n \to W_e^n$.

Definition 5: $\mathbf{G}$ is locally W-passive if there exists $\varepsilon > 0$ such that $\langle u, \mathbf{G}u\rangle_{W,T} \geq 0$ whenever $u \in W_e^n$ and $T > 0$ are such that $\|u\|_{W,T} < \varepsilon$;[6] $\mathbf{G}$ is locally strictly W-passive if there exists $\eta > 0$ such that $\mathbf{G} - \eta\mathbf{I}$ is locally W-passive.

Assume that $(I + \mathbf{G})^{-1}: W_e^n \to W_e^n$ is well defined, and set $\mathbf{H} = (\mathbf{G} - I)(I + \mathbf{G})^{-1}$. Clearly [6, p. 216], $\gamma_W(\mathbf{H}) \leq 1$ if $\mathbf{G}$ is W-passive, and $\gamma_W(\mathbf{H}) < 1$ if $\mathbf{G}$ is W-s and W-strictly passive. The proof that these properties still hold if local gains and local passivity are considered (instead of "global" ones) is straightforward. Therefore, by a standard scheme equivalence [6] and by Theorem 3, one obtains the following result.

Theorem 4: Let us consider the standard closed-loop system in Fig. 1, where $\mathbf{G_1}$ and $\mathbf{G_2}: W_e^n \to W_e^n$ are such that this closed-loop system is well posed. If $\mathbf{G_1}$ is locally W-passive and if $\mathbf{G_2}$ is l-W-s and locally strictly W-passive, then this closed-loop system is l-W-s.

The following result is obvious.

Proposition 7: Let $\mathbf{G}: W_e^n \to W_e^n$ be a LTI operator; then $\mathbf{G}$ is locally (strictly) W-passive iff $\mathbf{G}$ is (strictly) W-passive, and this condition is satisfied if $\mathbf{G}$ is (strictly) L_2-passive.

Therefore, the W-passivity or the strict W-passivity of a LTI operator can be analyzed in the frequency domain [6].

Let us now consider the case of a nonlinear memoryless time-invariant operator $\mathbf{G}$, defined as

$$(\mathbf{G}u)(t) = \Phi(u(t)), \quad \forall\, t \geq 0 \qquad (12)$$

[6] Hence an operator $\mathbf{G}: W^n \to W^n$ is locally W-passive iff $\inf_{\varepsilon>0} \inf_{0<\|u\|_W<\varepsilon} < u, \mathbf{G}u >_W \geq 0$ (by causality [19]).

where $\Phi: \Omega \to \mathbf{R}^n$ (Ω denoting some open neighborhood of zero in $\mathbf{R}^n$) is a C^1 function such that $\Phi(0) = 0$, and where u is any function in W_e^n, taking its values in Ω. Let $\mu(\cdot)$ denote the usual matrix measure [17], [6], i.e., for any matrix $A \in \mathbb{C}^{n\times n}$, $\mu(A) = \lambda_{\max}\ (A + A^*)/2$.

Proposition 8: Assume that $\mu(-\partial_u\Phi(0)) < 0$; then, $\mathbf{G}$ is locally strictly W-passive.

Proof: Let $\rho > 0$ be such that the ball $B_\rho = \{u: \|u\| < \rho\}$ is included in Ω; there exists a function $\alpha: \beta_\rho \to \mathbf{R}^n$ such that $\alpha(u)$ tends to zero as u tends to zero and $\Phi(u) = \partial_u\Phi(0)u + \alpha(u)\|u\|$ if $\|u\| < \rho$. Let $\beta = -\mu(-\partial_u\Phi(0)) > 0$; as $\partial_u\Phi$ is continuous, there exists $\varepsilon \in (0, \rho]$ such that $\|\alpha(u)\| < \beta/2$ and $\bar\sigma(\partial_u\Phi(u) - \partial_u\Phi(0)) < \beta/2$ if $\|u\| < \varepsilon$. Hence, for $\|u\| < \varepsilon$, one has $-u^T\Phi(u) - v^T\partial_u\Phi(u)v \le -(\beta/2)(\|u\|^2 + \|v\|^2)$, for any vector v in $\mathbf{R}^n$; therefore (by Proposition 1) if $u \in W_e^n$ is such that $\forall T > 0, \|u\|_{W,T} < \varepsilon$, one has $-\langle u, \mathbf{G}u\rangle_{W,T} \le -(\beta/2)(\|u\|_{W,T})^2$.

Remark 5: Assume that $\mathbf{G}$ is time-varying, i.e., satisfies (4), where Φ satisfies the conditions of Proposition 4. Generalizing the above rationale, it easy to prove that $\mathbf{G}$ is locally strictly W-passive if i) $\inf_{t\ge 0}\{-\mu[-\partial_u\Phi(t,0)]\} = \delta > 0$ and ii) $\beta < 2\delta$, where β is defined by (7).

Example 2: Let $\mathbf{G_1}$ be the LTI operator with transfer function $\frac{b}{1+ae^{-\tau s}+Ts}, |a| < 1, b > 0, T > 0$, and let $\mathbf{G_2}$ be the operator $\mathbf{G}$ defined by (4), with $\Phi(t,u) = (\delta+1+\cos\omega t)\sin u$; $\mathbf{G_1}$ is W-passive and $\mathbf{G_2}$ is l-W-s and locally strictly W-passive if $0 \le \omega < 2\delta$ (by Remark 5); hence, by Theorem 4, the closed-loop system in Fig. 1 is l-W-s if the latter condition is satisfied.

B. Local Sector Conditions

Other standard loop transformations can be made to obtain local sector conditions for local W-stablity from Theorem 3 (see e.g., [17, Section 6.6.1]). Let us consider the closed-loop system in Fig. 1 with $n = m$ and assume that it is well posed. Let r and c be reals such that r is nonzero and $(I_n + c\mathbf{G_1})^{-1}: W_e^n \to W_e^n$ is well defined and causal. According to these standard transformations and to Theorem 3, a sufficient condition for this closed-loop system to be locally W-stable is

$$\gamma_{Wl}\big(r\mathbf{G_1}(I_n + c\mathbf{G_1})^{-1}\big) \le 1 \tag{13}$$

$$\gamma_{Wl}\big((\mathbf{G_2} - cI_n)r^{-1}\big) < 1. \tag{14}$$

Now, set $a = c - r$ and $b = c + r$. It is easy to prove that (13) is satisfied if

$$\exists\, \varepsilon > 0: \|u\|_{W,T} < \varepsilon \Rightarrow \langle (I_n + a\mathbf{G_1})u, (I_n + b\mathbf{G_1})u\rangle_{W,T} \ge 0. \tag{15}$$

Similarly, (14) is satisfied if

$$\exists\, \varepsilon > 0, \exists\, \eta > 0: \|u\|_{W,T} < \varepsilon \Rightarrow \langle (\mathbf{G_2} - aI_n)u, (\mathbf{G_2} - bI_n)u\rangle_{W,T} \le -\eta(\|u\|_{W,T})^2. \tag{16}$$

Definition 6: If condition (16) is satisfied, $\mathbf{G_2}$ is said to be locally strictly inside the W-sector $[a, b]$.

In case where $\mathbf{G_1}$ is a LTI operator with transfer matrix G_1, one has $\langle (I_n + a\mathbf{G_1})u,\ (I_n + b\mathbf{G_1})u\rangle_{W,T} \ge 0$ if $\langle (I_n + a\mathbf{G_1})u, (I_n + b\mathbf{G_1})u\rangle_{2,T} \ge 0$; therefore, in the case $n = 1$, condition (15) can be transformed into a "circle condition" for G_1 [20]; a multivariable version of this condition can also be formulated, a and b being matrices, or even LTI operators satisfying some additional conditions [13], [12].

Now, let us consider the case where $\mathbf{G_2}$ is a nonlinear memoryless time-invariant operator of the form (12), where $\Phi: \Omega \to \mathbb{R}^n$ (Ω denoting some open neighborhood of zero in $\mathbb{R}^n$) is a C^1 function such that $\Phi(0) = 0$. It is easy to prove that $\mathbf{G_2}$ is locally strictly inside the W-sector $[a, b]$ if

$$\mu[(\partial_u\Phi(0) - aI_n)^T(\partial_u\Phi(0) - bI_n)] < 0.$$

The time-varying case can also be considered (like in Remark 5), but is slightly more complicated.

Example 3: Consider the closed-loop system in Fig. 1, where $\mathbf{G_1}$ is the LTI operator with transfer function $G_1(s) = [e^{-0.1s} + \frac{2}{s-1}][1 + e^{-0.1s} - \frac{2}{s+4}]$ and where $\mathbf{G_2}$ is an operator of the form (12), where $\Phi(u) = k\ tg\ u, k$ being any real such that $a < k < b$, with $a = 0.667$ and $b = 0.95$. The Nyquist plot of $G_1(i\omega)$ does not intersect the interior of the disk $D(a, b)$, centered on the real axis and whose circumference passes through the two points $-1/a$ and $-1/b$; moreover, this plot encircles $D(a, b)$ one time in the counterclockwise sense (and note that this number is equal to the number of poles of $G_1(s)$ with strictly positive real part): see [17, Section 6.6, Example 56]. Now, $\mathbf{G_2}$ is locally strictly inside the W-sector $[a, b]$. Therefore, the closed-loop system is l-W-s.

VII. Conclusion

For several reasons, W-stability and local W-stability are useful in system theory. Roughly speaking, a system is l-W-s if, when excited by a signal with small energy and filtered by a first order low-pass filter $\mathbf{K}_\tau$, its output, "filtered" by the inverse filter, is of small energy (Section II). Therefore, this type of stability is of significance in cases where disturbances, reference signals, etc., acting on the system can be considered as filtered signals (all signals are probably of this kind in practice). Moreover, W-stability and asymptotic stability are closely related in case of a minimal realization (Section IV). The local-W-gain of a time-invariant nonlinear system is nothing but the H_∞-norm of the transfer matrix of its linear approximation (Theorem 1), so this notion is clear. W-stability is well suited to formulate a useful local small gain theorem (Remark 4).

An interesting possible application of the local stability results established in this paper is the robustness study of LTI systems perturbed by nonlinearities which are only locally bounded [5]. Local input–output stability results can also be obtained in the discrete-time case [2].

References

[1] V. Alexeev, V. Tikhomirov, and S. Fomine, *Commande Optimale.* Moscow: Mir, 1982.

[2] H. Bourlès, "A local small gain theorem for discrete-time systems," in *Proc. 33rd Conf. Dec. Contr.*, Lake Buena Vista, FL, Dec. 14–16, 1994, pp. 2137–2138.

[3] H. Bourlès and F. Colledani, "W-stability and local small gain theorem," in *Proc. IFAC Conf. System, Structure and Control*, Nantes (France), July 5–7, 1995.

[4] M. J. Chen and C. A. Desoer, "Necessary and sufficient condition for robust stability of linear distributed feedback systems," *Int. J. Contr.*, vol. 35, pp. 255–267, 1982.

[5] F. Colledani, H. Bourlés, and M. P. Vanhersecke, "Robust controller with local linearizing feedback for frequency/power control of power plants," in *Proc. 32nd Conf. Dec. Contr.*, San Antonio, TX, Dec. 13–17, 1993, pp. 3740–3741.

[6] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties.* New York: Academic, 1975.

[7] B. A. Francis, *A Course in H_∞ Control Theory.* Berlin: Springer Verlag, 1987.

[8] D. J. Hill and P. J. Moylan, "Connections between finite-gain and asymptotic stability," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 931–936, 1982.

[9] Z. P. Jiang, L. Praly, and A. R. Teel, "Propagating the input-to-state stability property through integrators and applications," submitted for publication in M.C.S.S.

[10] A. Kolmogorov and S. Fomine, *Eléments de la Théorie des Fonctions et de l'analyse Lonctionnelle.* Moscow: Mir, 1977.

[11] D.-J. Pan, Z.-Z. Han, and Z.-J. Zhang, "Bounded-input—Bounded-output stabilization of nonlinear systems using state detectors," *Syst. Contr. Lett.*, vol. 21, pp. 189–198, 1993.

[12] M. G. Safonov, *Stability and Robustness of Multivariable Feedback Systems.* Cambridge, MA: MIT Press, 1980.

[13] M. G. Safonov and M. Athans, "A multiloop generalization of the circle criterion for stability margin analysis," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 415–422, 1981.

[14] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 435–443, 1989.

[15] ——, "Further facts about input-to-state stabilization," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 473–476, 1990.

[16] F. Trèves, *Topological Vector Spaces, Distributions and Kernels.* London: Academic, 1967.

[17] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[18] M. Vidyasagar and A. Vannelli, "New relationships between input-output and Lyapunov stability," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 481–483, 1982.

[19] J. C. Willems, *The Analysis of Feedback Systems.* Cambridge, MA: MIT Press, 1971.

[20] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 228–238 (Part I), pp. 465–476 (Part II), 1966.

The Analysis of Eigenvalue Assignment Robustness

Yuguang Fang

***Abstract*—In this short note, we have shown that the results obtained recently in [1] are conservative. A new generalization which can overcome such conservatism is presented.**

In [1], Wang and Lin studied the robust eigenvalue assignment for systems with parameter perturbations via matrix measures. They used three special matrix measures to obtain testable conditions for robust eigenvalue assignment. In this short note, we will show how conservative their results are and derive a new generalization which overcomes the conservative nature of their approach.

To facilitate the discussion, we first give the general concept of a matrix measure. Let C^n (R^n) denote all ordered n-tuples of complex (real) numbers and $C^{n\times n}$ ($R^{n\times n}$) represent the set of all $n \times n$ matrices with complex (real) entries. Let $\|\cdot\|$ be any vector norm on C^n and $\|A\|$ denote the matrix norm of A induced by the vector norm $\|\cdot\|$, the corresponding matrix measure of A induced by the vector norm $\|\cdot\|$ is defined by

$$\mu(A) = \lim_{\xi \downarrow 0^+} \frac{\|I + \xi A\| - 1}{\xi}.$$

Detailed properties can be found in [2]. As in [1], let H denote the complex left-half plane divided by a line L, which intersects with the real axis at $(\alpha, 0)$ and has an angle θ with the imaginary axis, i.e.,

$$H = \{z = x + jy \mid y < \alpha - (\tan\theta)x\}.$$

The following theorem is the main result obtained by Wang and Lin [1].

Theorem 1: All the eigenvalues of the matrix A are located in the region H if

$$\mu_p\left(e^{-j\theta}A\right) < \alpha\cos\theta$$

where $j = \sqrt{-1}$, μ_p denote the matrix measure induced by the p-norm and $p = 1$ or 2 or ∞.

The following example illustrates the conservative nature of Theorem 1.

Example 1: Choose H to be the left-half plane (i.e., $\theta = 0$ and $\alpha = 0$). Then Theorem 1 reduces to a stability test result. Let $A = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$. It is obvious that A is stable (or its eigenvalues are in H). Theorem 1, however, cannot be used to test the stability of A. In fact, according to Table I in [1], we have

$$\mu_1(A) = 1, \quad \mu_2(A) = 0, \quad \mu_\infty(A) = 1$$

thus the conditions in Theorem 1 can not be satisfied.

To overcome this problem, we generalize Theorem 1 to the following result.

Theorem 2: All the eigenvalues of the matrix A are located in the region H if there exists a matrix measure μ such that

$$\mu\left(e^{-j\theta}A\right) < \alpha\cos\theta.$$

To prove this, we only need the following property of a matrix measure.

Lemma: For any matrix measure μ, any complex number c and any matrix A, we have

$$\mu(A + cI) = \mu(A) + \Re(c)$$

where $\Re(c)$ denotes the real part of c.

Proof: If c is complex, let $c = \Re(c) + bj$, then we have $\mu(A + cI) = \mu(A + bjI) + \mathrm{Re}(c)$, where $j = \sqrt{-1}$. So it suffices to show that $\mu(A + bjI) = \mu(A)$. In fact

$$\begin{aligned}
&\mu(A + bjI) \\
&= \lim_{\theta\downarrow 0^+} \frac{\|I + \theta(A + bjI)\| - 1}{\theta} \\
&= \lim_{\theta\downarrow 0^+} \frac{|1 + b\theta j|\|I + \frac{\theta}{1+b\theta j}A\| - 1}{\theta} \\
&= \lim_{\theta\downarrow 0^+} \frac{\sqrt{1 + b^2\theta^2}\|I + \frac{\theta}{1+b^2\theta^2}A - \frac{bj}{1+b^2\theta^2}A\theta^2\| - 1}{\theta} \\
&= \lim_{\theta\downarrow 0^+} \frac{\sqrt{1 + b^2\theta^2}\|I + \frac{\theta}{1+b^2\theta^2}A\| - 1}{\theta} = \mu(A)
\end{aligned}$$

where we have used the fact that, as $\theta \downarrow 0^+$, $\theta/(1 + b^2\theta^2) \downarrow 0^+$. This completes the proof of Lemma. □

Manuscript received June 9, 1994; revised February 3, 1995. This work was supported in part by the Scientific Research Laboratories at Ford Motor Company, Dearborn, MI.

The author is with the Department of Electrical, Computer and Systems Engineering, Boston University, Boston, MA 02215 USA.

IEEE Log Number 9410785.

0018-9286/95$04.00 © 1995 IEEE