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Discussing some examples of linear system interconnections

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Abstract

The interconnection of linear systems may exhibit disturbing phenomena, like hidden modes or a lack of control variables. These features, which cannot be taken into account by the transfer function approach, are here explained by utilizing the language of modules, which is strongly related to Willems' behavioral approach. Several examples are carefully discussed.

Keywords: Linear systems; Interconnection; Feedback; Transfer functions; Behavioral approach; Controllability; Observability; Hidden modes; Modules; Principal ideal rings; Fibered sums and products

0. Introduction and motivating examples

Consider, as in [20] and [2], the constant linear monovariable system

$$\ddot{y} - y = \dot{u} - u, \tag{1}$$

the transfer function of which is $(s-1)/(s^2-1) = 1/(s+1)$. It corresponds to the block diagram in Fig. 1, since $\dot{u} - u = \dot{v} + v$, $\dot{y} - y = v$.

Set $z = y + \dot{y} - u$. Then, $\dot{z} - z = \ddot{y} - y - \dot{u} + u = 0$; z satisfies an unstable equation which corresponds to the hidden mode 1 of (1). It clearly implies that system (1) is not stabilizable.

Take now the "reverse" block diagram as in Fig. 2., which corresponds to $u = \dot{w} - w = y + \dot{y}$, therefore, to

$$\dot{y} + y = u. \tag{2}$$

Its transfer function is again $1/(s+1)$; system (2) is input–output stable (see Section 1.10), hence it does not possess the same input–output behavior as (1). The internal variable w, which satisfies $\dot{w} - w = u$,

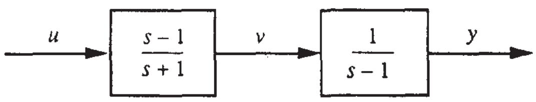

Fig. 1. System (1).

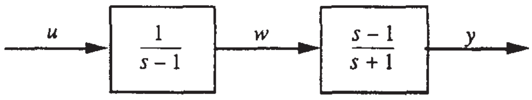

Fig. 2. System (2).

diverges when $\lim_{t\to\infty} u(t) = 0$: it again corresponds to the hidden mode 1.

Last, consider the feedback system (3) in Fig. 3.

Its transfer function is $T/(1+TS)$. If $TS = -1$, the loop becomes "ill-posed" in the sense of [29]. What is then the nature of system (3)?

These phenomena, which have been sometimes ignored in the control literature, seem to be difficult to explain in any classic framework. The difference between systems (1) and (2) cannot be taken into account by the transfer function approach, although it certainly could be within the polynomial approach [1, 19, 20, 27]. System (3), when $ST = -1$, is much

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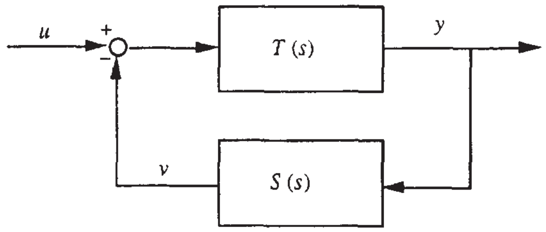

Fig. 3. Feedback system (3).

worse: Section 3.2 will demonstrate that the control variables, which are the free variables *par excellence*, loose their independence and must satisfy a homogeneous linear differential equation. Thus, a fresh look at the utmost important notion of *interconnection* confirms Willems' standpoint [31]: it is often misleading to distinguish between system variables. Interconnections, which have already been examined within the *behavorial* setting [32], (see, also, [22]), is here expressed via a standard algebraic tool, i.e., *fibered sums* of modules [23, 24]. All phenomena described above are then most easily understood.

The paper is organized as follows. The module-theoretic approach is first briefly reviewed. Section 2 relates interconnection to the fibered sum with some examples. Section 3 examines the above case-studies and another one, slightly different from Fig. 3, which displays a surprising change of rank. An appendix demonstrates the equivalence with the behavorial approach to interconnection [32] by taking advantage of its recent categorical interpretation [7] (see, also, [26]).

A preliminary version has already been published in [15].

1. A brief overview of the module-theoretic language

(See also [16])

1.1. Let k be a given differential ground field [21], where the derivation is written $\mathrm{d}/\mathrm{d}t =$ "$\cdot$". A *constant* $c \in k$ is an element such that $\dot{c} = 0$. A *field of constants* only contains constant elements.

1.2. Write $k[\mathrm{d}/\mathrm{d}t]$ the integral [1] ring of linear differential operators of the form

$$\sum_{\text{finite}} a_\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}, \quad a_\alpha \in k.$$

[1] By *integral*, we mean without "zero divisors".

This ring, which is commutative if, and only if, k is a field of constants, still is a left and right principal ideal ring [5].

1.3. All modules considered here are finitely generated left $k[\mathrm{d}/\mathrm{d}t]$-modules.

Notation. The module spanned by the set $w = \{w_1, \ldots, w_q\}$ is written $[w]$.

1.4. An element m of a module M is *torsion*, if, and only if, there exists $\pi \in k[\mathrm{d}/\mathrm{d}t]$, $\pi \neq 0$, such that $\pi m = 0$. A *torsion module* only contains torsion elements. The set of all torsion elements of a module is a submodule, called the *torsion submodule*. The next result is well known [5]:

Theorem. *A module is torsion if, and only if, its dimension as a k-vector space is finite.*

1.5. A module is free if, and only if, its torsion submodule is trivial, i.e., equal to $\{0\}$ [2]. The following theorem, which is classic [5], plays a crucial role.

Theorem. *A module M can be written as a direct sum $M = T \oplus \Phi$, where T is its torsion submodule and Φ a free module, which is unique up to isomorphism.*

1.6. The *rank* of a module M, which is written $rk(M)$, is the rank of the free module Φ: it is equal to the cardinality of any basis of Φ. A module is torsion if, and only if, its rank is zero.

1.7. A (*linear*) *system* is a module [10,12]. A (*linear*) *dynamics* [10] $\mathscr{D}$ is a system where we distinguish a finite set $u = \{u_1, \ldots, u_m\}$ of *input* variables such that the quotient module $D/[u]$ is torsion. The input u is *independent* if, and only if, the module $[u]$ is free of rank m. We may also distinguish a finite set $y = \{y_1, \ldots, y_p\}$, called the *output*. See [10] for the connection with state-variable representations.

1.8. A system Λ is *controllable* [10, 12] if, and only if, the module Λ is free. A dynamics $\mathscr{D}$ with input u and output y is *observable* [10] if, and only if, the two modules $\mathscr{D}$ and $[u, y]$ coincide ($[u, y]$ denotes the module spanned by the elements of u and y).

[2] This characterization of free modules is valid for finitely generated modules over principal ideal rings, where any torsion-free module is free [5].

1.9. Assume that k is a field of constants, the field $\mathbb{R}$ of real numbers for instance. *Hidden modes*, or *decoupling zeros*, are related to a lack of controllability and/or observability [3, 11, 19, 28].

The derivation $\mathrm{d}/\mathrm{d}t$ induces a k-linear endomorphism $\tau : T \to T$ of the torsion submodule [3] of a system Λ. The *input decoupling zeros* are the eigenvalues of τ over an algebraic closure $\bar{k}$ of k [3, 11].

Similarly, $\mathrm{d}/\mathrm{d}t$ induces a k-linear endomorphism $\sigma : \mathscr{D}/[u, y] \to \mathscr{D}/[u, y]$ of the quotient module $\mathscr{D}/[u, y]$ which is torsion. The *output decoupling zeros* are the eigenvalues of σ over $\bar{k}$ [3, 11].

1.10. Assume now, that k is the field $\mathbb{R}$ of real numbers. The derivation $k[\mathrm{d}/\mathrm{d}t]$ induces an $\mathbb{R}$-linear endomorphism $\mu : \mathscr{D}/[u] \to \mathscr{D}/[u]$ of the torsion module $\mathscr{D}/[u]$. The *poles* (or *system poles*) of $\mathscr{D}$ are the eigenvalues of μ over the field $\mathbb{C}$ of complex numbers. The dynamics $\mathscr{D}$ is said to be *internally stable* if, and only if, the real parts of its poles are strictly negative. Choose an output $y = (y_1, \ldots, y_p)$. The dynamics $\mathscr{D}$ is said to be *input–output stable* if, and only if, the poles of $[y, u]$, viewed as a dynamics with input u, have strictly negative real parts.

The poles of system (1), for instance, are 1 and -1. It is not internally stable. Since it is observable, it is neither input–output stable. System (2), the poles of which are also 1 and -1, is again not internally stable. But the only pole of $[u, y]$ is -1, hence system (2) is input–output stable.

1.11. The ring $k[\mathrm{d}/\mathrm{d}t]$ verifies the Ore property [5], and therefore possesses a skew quotient field $k(\mathrm{d}/\mathrm{d}t)$. The *transfer vector space* [14] of a system Λ is the tensor product $\hat{\Lambda} = k(\mathrm{d}/\mathrm{d}t) \otimes \Lambda$ which is a left $k(\mathrm{d}/\mathrm{d}t)$-vector space. The rank of Λ and the dimension of $\hat{\Lambda}$ coincide. The *transfer matrix* [14] is related to $\hat{\Lambda}$. Take two systems Λ_1, Λ_2 such that $\Lambda_1 \subset \Lambda_2$; then $\hat{\Lambda}_1$ coincides with $\hat{\Lambda}_2$ if, and only if, the quotient module Λ_2/Λ_1 is torsion.

2. Fibered sums and interconnections

2.1. Consider a family of modules M_α, $\alpha \in A$. Let E be a given module such that, for any $\alpha \in A$, there exists a morphism $h_\alpha : E \to M_\alpha$. Define the submodule $\mathscr{E}$ of the cartesian product $\times_{\alpha \in A} M_\alpha$ as being spanned by the elements of the form $(\ldots, 0, \ldots, h_{\alpha_1}(e), \ldots, 0, \ldots, -h_{\alpha_2}(e), \ldots, 0, \ldots), e \in E$, $\alpha_1 \neq \alpha_2$. The quotient module $\times_{\alpha \in A} M_\alpha / \mathscr{E}$ is called the *fibered sum* (or the *amalgamated sum*, or the *coproduct*) of the M_α's (see, e.g., [23, 24]). It is written $\coprod_{\alpha \in A, E} M_\alpha$.

[3] According to Section 1.4, this module is a finite-dimensional k-vector space.

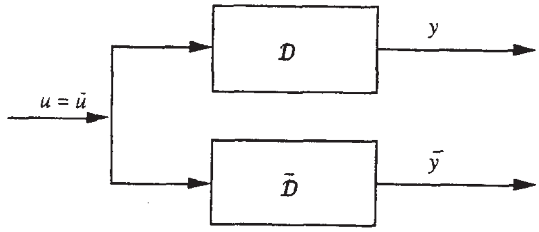

Fig. 4. Parallel interconnection.

2.2. When the M_α's are regarded as linear systems, the above fibered sum is called a (*system*) *interconnection*. Note that, like in the behavioral approach [7, 32], interconnections are defined without distinguishing between system variables.

2.3. Consider two dynamics, $\mathscr{D}, \bar{\mathscr{D}}$ with inputs $u = \{u_1, \ldots, u_m\}$, $\bar{u} = \{\bar{u}_1, \ldots, \bar{u}_{\bar{m}}\}$, and outputs $y = \{y_1, \ldots, y_p\}$, $\bar{y} = \{\bar{y}_1, \ldots, \bar{y}_{\bar{p}}\}$. Assume that $m = \bar{m}$, $u = \bar{u}$, and consider the usual *parallel interconnection* in Fig. 4.

Consider the free module $[\delta] = [\delta_1, \ldots, \delta_m]$ of rank m and the two canonical isomorphisms $\varphi : [\delta] \to [u], \delta_s \to u_s, s = 1, \ldots, m, \bar{\varphi} : [\delta] \to [\bar{u}], \delta_s \to \bar{u}_s$. $\mathscr{E}$ is the submodule of $\mathscr{D} \times \bar{\mathscr{D}}$ spanned by the elements of the form $(\varphi(\delta_s), -\bar{\varphi}(\delta_s))$, $s = 1, \ldots, m$. The above parallel interconnection is represented by the corresponding fibered sum which, for the sake of simplicity, is written $\mathscr{D} \coprod_{u=\bar{u}} \bar{\mathscr{D}}$. This module, in practice, is defined by the sets of equations defining $\mathscr{D}$ and $\bar{\mathscr{D}}$, plus the equation $u = \bar{u}$.

We might choose as an output of the parallel interconnection any k-linear combination of the components of y and $\bar{y}$.

2.4. Assume that $p = \bar{m}$, $y = \bar{u}$. Consider the usual *series interconnection* in Fig. 5.

Consider the free module $[\varepsilon] = [\varepsilon_1, \ldots, \varepsilon_p]$ of rank p and the two canonical epimorphisms $\phi : [\varepsilon] \to [y], \varepsilon_s \to y_s$, $s = 1, \ldots, p, \bar{\phi} : [\varepsilon] \to [\bar{u}], \varepsilon_s \to \bar{u}_s$. The above series interconnection is represented by the corresponding fibered sum which, for the sake of simplicity, is written $\mathscr{D} \coprod_{y=\bar{u}} \bar{\mathscr{D}}$. This module is defined

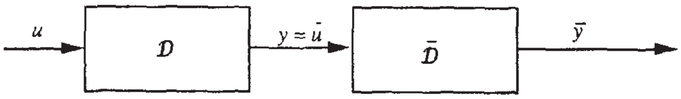

Fig. 5. Series interconnection.

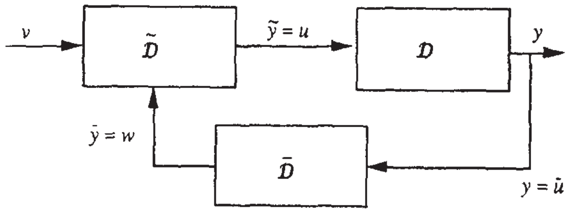

Fig. 6. Feedback interconnection.

by the sets of equations defining $\mathscr{D}$ and $\bar{\mathscr{D}}$, plus the equation $y = \bar{u}$[4].

2.5. Examples. The series interconnection in Fig. 1 (resp. in Fig. 2) is defined by $\dot{u} - u = \dot{v} + v$, $v = \dot{y} - y$ (resp. $u = \dot{w} - w = y + \dot{y}$).

2.6. Take a third dynamics $\tilde{\mathscr{D}}$ with input $\tilde{u} = \{\tilde{u}_1, \ldots, \tilde{u}_{\tilde{m}}\}$ and output $\tilde{y} = \{\tilde{y}_1, \ldots, \tilde{y}_{\tilde{p}}\}$. Consider the classic feedback interconnection in Fig. 6.

The input $\tilde{u} = v \cup w$ is divided into two parts. Set $\tilde{y} = u$, $y = \bar{u}$, $\bar{y} = w$. Following the notations of Sections 2.3 and 2.4, the above block diagram corresponds to the fibered sum $\coprod_{u=\tilde{y}, y=\bar{u}, \bar{y}=w}(\mathscr{D}, \bar{\mathscr{D}}, \tilde{\mathscr{D}})$.

3. Some phenomena

3.1. Lack of controllability and observability

3.1.1. Interconnecting controllable (resp. observable) linear systems may give rise to an uncontrollable (resp. unobservable) one (see, e.g., [4, 18] for calculations in some concrete situations). When $k = \mathbb{R}$, the corresponding hidden modes (see 1.9) may exhibit positive real parts which imply unstability. Uncontrollability and unobservability, which both correspond to torsion modules, cannot be detected by transfer functions (see 1.11).

3.1.2. Example. Consider in system (1) the variable $z = y + \dot{y} - u$, which is torsion since $\dot{z} - z = 0$. The system is uncontrollable and the corresponding input decoupling zero, which is equal to 1, is unstable (see Section 1.10).

3.1.3. Example. In system (2), the variable w, which verifies $\dot{w} - w = u = \dot{y} + y$, cannot be expressed as an $\mathbb{R}$-linear combination of u, y and of a finite number of their derivatives.[5] It means unobservability of system (2) (see [10]). The corresponding output decoupling zero, which is again 1, is also unstable (see Section 1.10).

3.1.4. Example. Consider system (3), and write $T(s) = a(s)/b(s)$, $S(s) = c(s)/d(s), a, b, c, d \in R[s]$, $abcd \neq 0$; a and b (resp. c and d) are coprime. The system is governed by the two homogeneous equations

$$
\begin{aligned}
&a\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(u - v) = b\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y, \\
&d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)v = c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y
\end{aligned}
\tag{3}
$$

(i) If $ac + bd \neq 0$, i.e., $ST \neq -1$, y and v can be calculated from u.

(ii) If $ac + bd = 0$, i.e., $ST = -1$, there exists a constant $\lambda \in \mathbb{R}, \lambda \neq 0$, such that $d = \lambda a, c = -\lambda b$. The input u must satisfy $a(\mathrm{d}/\mathrm{d}t)u = 0$ and becomes a torsion element: system (3) is ill-posed [29]. The remaining variables y and v, which are related by $a(\mathrm{d}/\mathrm{d}t)v + b(\mathrm{d}/\mathrm{d}t)y = 0$, span a free module of rank 1 (because a and b are coprime, cf. [30, 14]). This rather surprising lack of controllability concerns here the control variable.

[4] The variable $y = \bar{u}$ is assumed to have no physical meaning in the input–output setting of [2].

[5] The quotient module $[u, y, w]/[u, y]$ is the torsion module defined by $\dot{\underline{w}} - \underline{w} = 0$, where $\underline{w}$ is the canonical image of w.

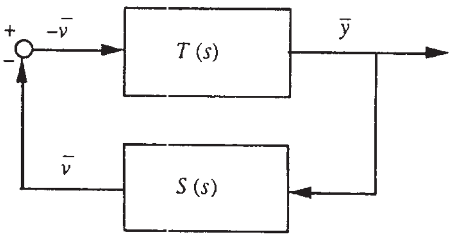

Fig. 7. Feedback system (4).

3.2. *Change of rank*

3.2.1. The rank of a system module should be understood as the maximum number of independent input channels (cf. [9]). This rank is usually clear from the physical context, but may exhibit some abrupt change in some peculiar interconnections.

3.2.2. Example. The feedback system (4) depicted in the block diagram in Fig. 7 differs from the one in Fig. 3 by the suppression of u. The transfer functions $T(s) = a(s)/b(s), S(s) = c(s)/d(s)$ are the same as in 3.1.4. System (4) is governed by the equations

$$\begin{aligned} -a\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{v} &= b\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{y}, \\ d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{v} &= c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{y} \end{aligned} \tag{4}$$

There exists a canonical isomorphism $[u, v, y]/[u] \to [\bar{v}, \bar{y}], v \to \bar{v}, y \to \bar{y}$, where u, v, y (resp. $\bar{v}, \bar{y}$) obey to (3) (resp. (4)). It implies the following conclusions:

(i) If $ac + bd \neq 0$, i.e., ST $\neq -1$, the module $[\bar{v}, \bar{y}]$ is torsion: $\bar{v}$ and $\bar{y}$ satisfy a linear homogeneous differential equation.

(ii) If $ac + bd = 0$, i.e., ST $= -1$, the module $[\bar{v}, \bar{y}]$ is free of rank 1: $\bar{v}$ and $\bar{y}$ are not determined by linear homogeneous differential equations. System (4) is again ill-posed: when assigning to $\bar{v}$ (resp. $\bar{y}$) an arbitrary value, i.e., an arbitrary function, $\bar{y}$ (resp. $\bar{v}$) can be calculated from (4).

4. Conclusion

Even in the general noncommutative case, the ring $k[\mathrm{d}/\mathrm{d}t]$ is left and right Euclidian (cf. [5]). This renders calculations in modules very similar to what is already known in the polynomial approach to linear time-invariant systems. It should therefore be possible to check a large-scale interconnection by *computer algebra*. Discrete time, as well as delays, may be treated along the same lines by employing the formulation of [13, 17, 25].

The converse problem of decomposing a given linear system into "simple" parts should also be considered. Symmetries (see [6, 8]) might be useful in that aspect.

Acknowledgements

The authors would like to thank G. Conte for making the paper [7] available to them before its publication.

Appendix: A bridge with the behavorial approach

A.1. For the sake of simplicity, we restrict ourselves, like in [12], to constant linear systems, i.e., we assume that $k = \mathbb{R}$. We first recall [12] how trajectories are related to a module-theoretic standpoint. Denote by $C^\infty(t_1, t_2)$ the set of C^∞-functions $(t_1, t_2) \to \mathbb{R}$, where $-\infty \leqslant t_1 \nleqslant t_2 \leqslant +\infty$; $C^\infty(t_1, t_2)$ possesses a canonical structure of $\mathbb{R}[\mathrm{d}/\mathrm{d}t]$-module. The trajectories on the time interval (t_1, t_2) of a constant linear system M, i.e., of a finitely generated $\mathbb{R}[\mathrm{d}/\mathrm{d}t]$-module M, is the set $\mathrm{Hom}(M, C^\infty(t_1, t_2))$ of morphisms between the two $\mathbb{R}[\mathrm{d}/\mathrm{d}t]$-modules M and $C^\infty(t_1, t_2)$; $\mathrm{Hom}(M, C^\infty(t_1, t_2))$ also possesses a canonical structure of an $\mathbb{R}[\mathrm{d}/\mathrm{d}t]$-module.

A.2. Let $f : M_1 \to M_2$ be an $\mathbb{R}[\mathrm{d}/\mathrm{d}t]$-module morphism between the two systems M_1 and M_2. It yields an $\mathbb{R}[\mathrm{d}/\mathrm{d}t]$-module morphism

$$f^\# : \mathrm{Hom}(M_2, C^\infty(t_1, t_2)) \to \mathrm{Hom}(M_1, C^\infty(t_1, t_2))$$

defined by

$$\forall \tau \in \mathrm{Hom}(M_2, C^\infty(t_1, t_2)), \quad f^\# \tau = \tau f. \tag{A.1}$$

One easily verifies that if f is injective (resp. surjective), $f^\#$ is surjective (resp. injective).

A.3. We now need some elementary facts from *category* theory (see, e.g., [23,24]). A category $\mathscr{C}$ is a collection of *objects*. Between each pair A, B of objects in $\mathscr{C}$, there is a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$, or $\mathrm{Hom}(A, B)$, of *$\mathscr{C}$-morphisms*. In the category of $\mathbb{R}[\mathrm{d}/\mathrm{d}t]$-modules, for instance, the morphisms are the $\mathbb{R}[\mathrm{d}/\mathrm{d}t]$-linear mappings.

A.4. A *contravariant functor* F between two categories $\mathscr{C}$ and $\mathscr{D}$ assigns to each object A in $\mathscr{C}$ an object $F(A)$ and to each $\mathscr{C}$-morphism $\varphi : A \rightarrow B$ a $\mathscr{D}$-morphism $F(\varphi) : F(B) \rightarrow F(A)$. Note the reversed order.

Example. Consider $\mathrm{Hom}(\bullet, C^{\infty}(t_1,t_2))$, where the point $\bullet$ denotes an arbitrary finitely generated $\mathbb{R}[\mathrm{d}/\mathrm{d}t]$-module. It is a contravariant functor since it associates to each finitely generated $\mathbb{R}[\mathrm{d}/\mathrm{d}t]$-module M the $\mathbb{R}[\mathrm{d}/\mathrm{d}t]$-module $\mathrm{Hom}(M, C^{\infty}(t_1,t_2))$ and to each morphism $M_1 \rightarrow M_2$ a morphism $\mathrm{Hom}(M_2, C^{\infty}(t_1,t_2)) \rightarrow \mathrm{Hom}(M_1, C^{\infty}(t_1,t_2))$ via (A.1).

The *fibered product* may be defined as the fibered sum in Section 2.1 by reversing the arrows. The image of the fibered sum (resp. product) under a contravariant functor is a fibered product (resp. sum).

A.5. Take, as in 2.1, the fibered sum $M_1 \coprod M_2$ of two systems with respect to the morphisms $h_i : E \rightarrow M_i$, $i = 1,2$. Since the functor $\mathrm{Hom}(\bullet, C^{\infty}(t_1,t_2))$ is contravariant, we know from Section A.4 that

$$\mathrm{Hom}\,(M_1 \textstyle\coprod M_2, C^{\infty}(t_1,t_2))$$

is the fibered product of $\mathrm{Hom}(M_1, C^{\infty}(t_1,t_2))$ and $\mathrm{Hom}(M_2, C^{\infty}(t_1,t_2))$ with respect to the morphisms

$$h_i^{\#} : \mathrm{Hom}(M_i, C^{\infty}(t_1,t_2)) \rightarrow \mathrm{Hom}(E, C^{\infty}(t_1,t_2)),$$
$$i = 1,2.$$

We thus recover the behavorial interpretation of interconnection given in [7].

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