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Finite poles and zeros of linear systems: an intrinsic approach

HENRI BOURLÈS† and MICHEL FLIESS‡

Several kinds of finite poles and zeros are defined and their relationships are studied in an intrinsic manner. Some of these definitions are more precise than the usual ones (all kinds of multiplicities of an input–output decoupling zero are defined here). In addition, we believe that our approach clarifies those notions, and it makes it possible to very concisely establish fundamental relationships between various poles and zeros. Some new relationships are derived. The notions of input–output stability and transfer stability are also introduced. Several examples illustrate the theory.

1. Introduction

Numerous papers have been written on poles and zeros of linear systems. This subject is still topical, as shown by recent contributions (Schrader and Sain 1989, 1991, 1993, Schrader 1992, Schrader *et al.* 1995, Giust *et al.* 1993). Efficient algorithms have been developed for computing critical transmission poles and zeros of large scale systems, such as power systems (Martins 1986, Pérez-Arriaga *et al.* 1990, Martins *et al.* 1991) which can have up to 20 000 states (Rogers 1996). However, some theoretical points are not yet completely clear. Consider, as in Kailath (1980), the following example.

Example 1: The system considered here is defined by the polynomial matrix description (PMD)

$$D(s)\xi = N(s)u \tag{1}$$

$$y = Q(s)\xi + W(s)u \tag{2}$$

where s means $\mathrm{d}/\mathrm{d}t$ and the matrices $D(s), N(s), Q(s)$ and $W(s)$ are

$$D(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s^2(s+1) & s(s+2) \\ 0 & 0 & 0 & s+2 \end{bmatrix}, \quad N(s) = \begin{bmatrix} 0 \\ 0 \\ -s \\ 1 \end{bmatrix} \tag{3}$$

$$Q(s) = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix}, \quad W(s) = 0 \tag{4}$$

Assume that one wants to calculate, say, the input–output decoupling zeros of this

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system. A well-defined algorithm exists for this (Kailath 1980). However, notion of input–output decoupling zero is usually defined as the result of this algorithm; namely (and roughly speaking) they are the input-decoupling zeros which disappear when the unobservable part of the system is extracted; in an equivalent manner, they are the output-decoupling zeros which disappear when the uncontrollable part is extracted. This definition is rather fuzzy, and in the authors' opinion it is clearer and more convenient first to give a precise and concise definition of input–output decoupling zeros (and of all kinds of zeros and poles as well) and then to deduce from that definition an algorithm for calculating them. This is one of the aims of this paper (in addition, it will be shown that the various kinds of multiplicities of an input–output decoupling zero can be specified; see Remark 8 below).

The second aim is to define and study poles and zeros in an intrinsic approach. Modern control theory began in the early 1960s with the fundamental contributions of Kalman on the state-space representation of systems. A linear system was defined by the well-known four matrices of the Kalman representation. Wonham (1984) showed that, instead of reasoning with matrices, it is often more suitable to do it more intrinsically with the linear mappings, whose matrices are the representations in particular bases, or with spaces associated with those mappings (image, kernel, etc.). However, in many cases, the system equations are not naturally in state-space form, and the general description of a linear system is the Rosenbrock polynomial matrix description (1) and (2). However, the polynomial matrices $D(s), N(s), Q(s)$ and $W(s)$ are only representations of linear mappings in specific bases. As the entries of these matrices are elements of a ring (namely, the ring $\mathbf{R} = \mathbb{R}[s]$ of polynomials with real coefficients and indeterminate s) those mappings are morphisms of $\mathbf{R}$-modules (instead of morphisms of $\mathbb{R}$-vector spaces), and the spaces associated with them are $\mathbf{R}$-modules. The point is that a same system can be described by infinitely many PMDs, whereas the associated module is unique (up to isomorphism). More specifically, two PMDs are strictly equivalent (Rosenbrock 1970, Fuhrmann 1977); in other words, they can be considered as representing the same system (Perbeno 1977) if and only if (iff) they are associated with the same module (up to isomorphism). See Remark 1 below for more details. Therefore, a linear system is defined as being a module in this paper (in a mathematical point of view) because there is a one-to-one correspondence between a linear system (viewed as the equivalence class of all strictly equivalent PMDs which describe it) and the associated module.

This module theoretic setting, which was developed by Fliess (1990, 1991, 1992 a) (see also Fliess and Glad (1993), where the nonlinear case is treated as well) can be viewed as an abstract framework for manipulating the system equations. To be clear, let us take a very simple example (for an overview of basic notions in module theory, see below).

Example 2: The system considered here is defined by the PMD (1) and (2) with

$$D(s) = s^2(s-1)(s+2)^2, \quad N(s) = s(s-1)(s+1) \tag{5}$$

$$Q(s) = s^2, \quad W(s) = 0 \tag{6}$$

The module associated with (1) can be explicitly constructed as follows: let ε be a free $\mathbf{R}$-module of rank 1, and let $\{\varepsilon\}$ be a basis of ε. (Recall that a module is free if and only if it has a basis. A vector space is an example of a free module. Let M be a

free **R**-module; as in the case of a vector space, a subset $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is a basis of M if and only if $\varepsilon_1, \ldots, \varepsilon_n$ are **R**-linearly independent, and every element of M can be expressed as a **R**-linear combination of $\varepsilon_1, \ldots, \varepsilon_n$. This integer n depends only on M and is called the rank of M.) Let $\mathcal{F}$ be another free module, of rank 2, and let $\{\xi, \boldsymbol{u}\}$ be a basis of $\mathcal{F}$. Finally, let $f : \mathcal{E} \to \mathcal{F}$ be the **R**-morphism defined by $f(\varepsilon) = D(s)\xi - N(s)\boldsymbol{u}$.

Set $e = f(\varepsilon)$, so that $D(s)\xi - N(s)\boldsymbol{u} = e$, and let $[e]$ denote the submodule of $\mathcal{F}$ generated by e. Note that ξ and $\boldsymbol{u}$ do not satisfy (1); as they are free variables, they only satisfy the trivial equations $0\xi = 0, 0\boldsymbol{u} = 0$. The element e can be viewed as the deviation, or the error $D(s)\xi - N(s)\boldsymbol{u}$.

The module associated with (1) is the quotient module $\mathcal{F}/[e]$ (i.e. $\mathcal{F}/\mathrm{Im} f := \mathrm{coker} f$). Denote by ξ and u the images of ξ and $\boldsymbol{u}$ by the canonical epimorphism $\mathcal{F} \to \mathcal{F}/[e]$ those variables obviously satisfy (1). Roughly speaking, indeed, projecting in the quotient module $\mathcal{F}/[e]$ consists of replacing e by 0 in all equations involving the elements of $\mathcal{F}$, because the image of e by this projection is 0. Our module $\mathcal{F}/[e]$ is the module generated by ξ and u and is denoted by $[\xi, u]$ By definition, this module consists of all elements of the form $\alpha(s)\xi + \beta(s)u$, where $\alpha(s)$ and $\beta(s)$ belong to **R**. Equation (2) means that y belongs to $[\xi, u]$

The basic difference between modules and vector spaces can easily be understood through this example. Set $z = s(s+2)^2\xi - (s+1)u$. One has $\gamma z = 0$ where γ is the polynomial $\gamma(s) = s(s-1)$; although this polynomial is non-zero, this does not imply $z = 0$ (because γ is not invertible); z only satisfies the differential equation $s(s-1)z = 0$. This variable z is said to be a torsion element of the module $[\xi, u]$ It is seen below that the existence of such elements characterizes uncontrollable systems. Note that the morphism f is represented in the bases $\{\varepsilon\}, \{\xi, \boldsymbol{u}\}$ by the polynomial matrix $S(s) = [D(s) \;\; -N(s)]^{\mathrm{T}}$.

It was pointed out by Willems (1983, 1991) that in some cases, it may be misleading to distinguish between system variables. Set $w_1 = \xi, w_2 = u, w = [w_1 \;\; w_2]^{\mathrm{T}}$. The system (i.e. the module $[w]$ generated by w_1 and w_2 and constructed above) can be written

$$S^{\mathrm{T}}(s)w = 0 \tag{7}$$

The time functions that are the solutions of (7) on the real line $\mathbb{R}$ (or on an open interval of $\mathbb{R}$) are called the behaviour (Willems 1991) (Blomberg and Ylinen 1983). As stated above, we deal with the module generated by the abstract variable w and constructed as a quotient module; this abstract module is defined up to isomorphism. It has been shown above how this module is directly related to the equations. Our framework and that of Willems are closely connected (Fliess 1992 a, Fliess and Bourlès 1996).

Many kinds of pole and zero are defined in the literature. The most important among them are the transmission poles, the system poles, the blocking zeros, the transmission zeros, the decoupling zeros, the invariant zeros and the system zeros. The system poles and the transmission poles respectively govern the internal stability and the transfer stability. (If the initial conditions are not zero, the stability of all transmission poles, i.e. their location in the open left half-plane, is not sufficient to ensure the input–output stability of the system; see Bourlès (1994) and Definition 8 below.) The location of the blocking zeros plays a crucial role in the strong stabilization problem, i.e. the problem of stabilizing a system via a stable controller (Youla *et al.* 1974). The well known internal model principle (Wonham 1984)

consists in generating blocking zeros, and it is applicable only if the transmission zeros of the system satisfy a suitable condition; systems are more difficult to control when they have unstable transmission poles or zeros (Freudenberg and Looze 1988). The existence of hidden modes, i.e. of input- or output-decoupling zeros, is related to a lack of controllability or observability. The invariant zeros can be viewed as the system poles of the inverse system (when it exists) and are also important; for example, one of the basic assumptions of the well known Glover–Doyle algorithm in H_∞ theory is that two subsystems have no invariant zeros on the imaginary axis (Glover and Doyle 1988); see Kailath (1980), Schrader and Sain (1989) and Wonham (1984) for other uses of invariant zeros. Finally, the set of system zeros is, roughly speaking, the set of all zeros.

Most of these notions were defined, in the general framework of polynomial matrix descriptions (PMD) of linear systems, by Rosenbrock (1970, 1974), apart from the notions of invariant zero and of blocking zero, respectively, due to MacFarlane and Karkanias (1976) and to Ferreira and Bhattacharyya (1977). However, the topic of poles and zeros has been somewhat confused for many years, because certain authors used the same terminology to denote different notions. The situation was greatly clarified by the survey papers of MacFarlane and Karkanias (1976) and Schrader and Sain (1989), and by the textbooks of Kailath (1980) and Kaczorek (1992).

More recently, some authors have defined poles and zeros as modules (instead of numbers): see, for example Conte and Perdon (1985), Schrader (1992), Wyman and Sain (1987) and related references. In this more abstract approach, of course, poles and zeros contain more information, and this is useful in exploring, for instance, the question of multivariable pole–zero cancellations (Wyman and Sain 1981 b, Conte and Perdon 1986).

In this paper we define and study the modules associated with various kinds of poles and zeros, as in, for example Conte and Perdon (1985), Wyman and Sain (1981 a, 1987), and Schrader (1992). The latter are the Smith zeros of those modules (see Definition 2 below) and so they are numbers, as in Rosenbrock's standpoint.

A preliminary version has already been published by Bourlès and Fliess (1995).

2. Linear systems and dynamics

2.1. *A brief overview of module theory*

In what follows, $\mathbf{R}$ denotes the principal ideal domain $\mathbb{R}[s]$ and $\mathbf{F} = \mathbb{R}(s)$ denotes the quotient field of $\mathbf{R}$. Let $\boldsymbol{w} = \{w_1, \ldots, w_q\}$ be a finite subset of an $\mathbf{R}$-module M. The column matrix $[w_1 \ \cdots \ w_q]^{\mathrm{T}}$ and the submodule spanned by $\boldsymbol{w}$ are, respectively, written w and $[w]$ If $\boldsymbol{v}$ is another subset of M, $[w, v]$ denotes the submodule spanned by $\boldsymbol{w} \cup \boldsymbol{v}$. The module generated by the empty subset of M is the trivial submodule consisting of zero alone, and it is denoted by 0.

All the modules considered here are finitely generated $\mathbf{R}$-modules.

The properties of the modules recalled below are well known (MacLane and Birkhoff 1968).

An element m of a module M is torsion iff there exists a polynomial $\pi \in \mathbf{R}$, $\pi \neq 0$, such that $\pi m = 0$. A torsion module only contains torsion elements. The set of all torsion elements of a module is a submodule, called the torsion submodule. A module is free iff its torsion submodule is trivial. (This characterization of free

modules is valid for finitely generated modules over principal ideal domains, where any torsion free module is free.)

A module M can be written as a direct sum

$$M = T \oplus \Phi \tag{8}$$

where T is its torsion submodule and $\Phi \simeq M/T$ ($\simeq$ means is isomorphic to) is a free submodule, which is unique up to isomorphism.

The rank of a module M, which is written $rk(M)$, is the rank of the free submodule Φ and is equal to the cardinality of any basis of Φ. A module is torsion iff its rank is zero.

In this section, the construction made in the introduction (see Example 2) is generalized; this construction is classic (MacLane and Birkhoff 1968). For every module M there exists a short exact sequence (MacLane and Birkhoff 1968, Bourbaki 1961)

$$0 \to \varepsilon \xrightarrow{f} \mathcal{F} \xrightarrow{\phi} M \to 0 \tag{9}$$

where ε and $\mathcal{F}$ are free modules. This means that f is a monomorphism, that $\mathrm{Im} f = \ker \phi$ and that ϕ is an epimorphism. As a result, $M = \phi(\mathcal{F})$ is isomorphic to $\mathcal{F}/\mathrm{Im} f = \mathrm{coker} f$. The exact sequence (9) is called a presentation of M (Bourbaki 1961). (Note that one can consider a presentation of M where f is not a monomorphism. However, in this case, f can be replaced by its restriction f_1 to a submodule ε_1 of ε such that $\varepsilon = \ker f \oplus \varepsilon_1$. Then f_1 is a monomorphism and one has the short exact sequence (9), with f replaced by f_1 and ε by ε_1. This consists of removing the redundant equations (i.e. those which are **R**-linearly dependent on other ones).)

In the presentation (9), set $n = rk(\varepsilon), k = rk(\mathcal{F})$, and let $\alpha_i, 1 \le i \le r$, be the invariant factors of f (which are non-zero polynomials such that α_i divides α_{i+1}). One has $k \ge n, r = n$ and in the decomposition (8)

$$T \simeq \bigoplus_{i=1}^{r} \mathbf{R}/(\alpha_i), \quad \Phi \simeq \mathbf{R}^{k-n} \tag{10}$$

By (10), $rk(M) = k - n$. Thus the $\alpha_i, 1 \le i \le r$, are dependent only on M and are called the non-zero invariant factors of this module; in addition, M is said to have $k - n$ zero invariant factors.

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\mathrm{w} = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k\}$ be bases of ε and $\mathcal{F}$, respectively. In these bases, f is represented by a matrix $S(s)$; S is called a matrix of definition of M. The $\alpha_i, 1 \le i \le r$, are the invariant factors of S, and the Smith form of this matrix is

$$k \left\{ \underbrace{\begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_r \\ 0 & 0 & 0 \end{pmatrix}}_{r=n} \right.$$

As a result, one obtains in practice the decomposition (10) by calculating the Smith form of the matrix of definition $S(s)$.

Set $e_i = f(\varepsilon_i), 1 \le i \le n$, so that $e = S(s)^{\mathrm{T}} \boldsymbol{w}$, and let $w_i = \phi(\boldsymbol{w}_i), 1 \le i \le k$; then $M = [w] = [\boldsymbol{w}]/[e]$ and one has

$$S(s)^{\mathrm{T}} w = 0 \tag{11}$$

which is identical to (7).

Definition 1: Equation (11) is the equation of the module M in the chosen bases. □

Note that M has not an unique equation, but that any equation of M defines this module up to isomorphism.

In this section some mathematical results that will be useful below are recalled or established.

Let M be a module, and $\alpha_1, \ldots, \alpha_n$ be its non-zero invariant factors (i.e. the invariant factors of its torsion submodule). Set $\alpha = \alpha_1 \cdots \alpha_n$.

Definition 2: The Smith zeros of M are the roots of $\alpha(s)$ over the complex plane $\mathbb{C}$, including multiplicities. Now, consider a matrix $S(s)$, let $\alpha_1, \ldots, \alpha_n$ be its invariant factors, and set $\alpha = \alpha_1 \cdots \alpha_n$. The Smith zeros of S are the roots of $\alpha(s)$ over $\mathbb{C}$, including multiplicities (Kailath 1980, p. 580). Let z be a Smith zero (of a module M or of a matrix $S(s)$); the (non-zero) invariant factors are of the form $(s-z)^{\mu_1}\beta_1(s), \ldots, (s-z)^{\mu_n}\beta_n(s)$, where no $\beta_i(s), 1 \leq i \leq n$, is divisible by $s-z$. The integers $\mu_i, 1 \leq i \leq n$, are such that $0 \leq \mu_1 \leq \cdots \leq \mu_n$. The non-zero μ_i are called the structural indices (of M or of $S(s)$) at z, μ_n is called the order of z, and $\mu_1 + \cdots + \mu_n$ is called its degree. See § 3.9 for a dynamical interpretation of the order. □

This definition is taken by analogy with the structural indices of a rational matrix at a zero z, of the order of this zero, and of its degree (Kailath 1980, p. 447). By analogy with the multiplicities of an eigenvalue, μ can also be called the algebraic multiplicity of z, whereas the number of non-zero integers μ_i can be called its geometric multiplicity (Chen 1984, p. 41, footnote 14).

Obviously, the Smith zeros of a module M are the Smith zeros of any matrix of definition $S(s)$ of M, or of its transposate $S^{\mathrm{T}}(s)$. The Smith zeros of a morphism of free modules can be defined similarly.

Let M, M' and M'' be modules such that $M'' \subset M' \subset M$. Then (MacLane and Birkhoff 1968)

$$M/M' \simeq (M/M'')/(M'/M'')$$

Let M, M', M'', N' and N'' be modules such that $M = M' \oplus M''$, $N' \subset M'$ and $N'' \subset M''$. Then (MacLane and Birkhoff 1968)

$$(M' \oplus M'')/(N' \oplus N'') \simeq (M'/N') \oplus (M''/N'')$$

Lemma:

(*a*) *Let T_1 and T_2 be two torsion modules, with $T_2 \subset T_1$; one has*

$$\{\textit{Smith zeros of } T_1\} = \{\textit{Smith zeros of } T_2\} + \{\textit{Smith zeros of } T_1/T_2\}$$

(*b*) *Let T_1 and T_2 be two torsion submodules of a module M, such that $T_1 \cap T_2 = 0$. Then*

$$\{\textit{Smith zeros of } T_1 \oplus T_2\} = \{\textit{Smith zeros of } T_1\} + \{\textit{Smith zeros of } T_2\}$$

Proof: (*a*) Let $\alpha_1, \ldots, \alpha_r$ be the invariant factors of T_1. Write

$$T_1 = \bigoplus_{i=1}^{r} C_i$$

where $C_i \simeq \mathbf{R}/(\alpha_i)$. Then

$$T_2 = \bigoplus_{i=1}^{r} D_i$$

with $D_i = C_i \cap T_2$. One has $D_i \simeq \mathbf{R}/(\beta_i)$, where β_i divides α_i, i.e. $\alpha_i = \beta_i\gamma_i, \gamma_i \in \mathbf{R}$. Hence

$$T_1/T_2 \simeq \bigoplus_{i=1}^{r} \mathbf{R}/(\gamma_i)$$

Note that β_i divides β_{i+1}, but that γ_i does not necessarily divide γ_{i+1}, so that the γ_i are generally not the invariant factors of T_1/T_2. However, if $\delta_1, \ldots, \delta_s$ denote the invariant factors of T_1/T_2, one has $\delta_1 \cdots \delta_s = \gamma_1 \cdots \gamma_r$, and (a) is proved. The proof of (b) is similar. □

2.2. *Linear systems*

The discussion in the introduction (see Example 2) leads us to the following definition: a linear system is a module (Fliess 1990). (This module-theoretic setting should not be confused with the classic approach due to Kalman 1969; see also Fuhrmann 1981. Here modules encompass all system variables, whereas Kalman's modules are related to the state-variable representation.)

In this context, it may be useful to remember the following algebraic approach to Laplace transform and transfer matrices (Fliess 1994).

The transfer vector space of a system Λ is the tensor product $\hat{\Lambda} = \mathbf{F} \otimes \Lambda$ which is a $\mathbf{F}$-vector space. The Laplace transform of $\lambda \in \Lambda$ is $\hat{\lambda} = 1 \otimes \lambda \in \hat{\Lambda}$, where 1 is the unit element of $\mathbf{F}$. Roughly speaking, when dealing with Λ, the only multiplications allowed are multiplications by polynomials with indeterminate s, whereas when dealing with $\hat{\Lambda}$ multiplications by rational functions in s can be made (as with the usual Laplace transform, assuming that the initial conditions are zero).

Let us give in detail the meaning of this: roughly speaking, $\hat{\Lambda}$ consists of all elements of the form

$$\sum_{\text{finite}} g_i\lambda_i, \quad g_i \in \mathbf{F}, \quad \lambda_i \in \Lambda$$

Consider Example 2: one has (because the tensor product is $\mathbf{R}$-linear)

$$s^2(s-1)(s+2)^2\hat{\xi} = s(s-1)(s+1)\hat{u}$$

This equation can be multiplied by $1/s(s-1)$ (which belongs to $\mathbf{F}$) and one obtains

$$s(s+2)^2\hat{\xi} = (s+1)\hat{u}$$

(i.e. the common factor $s(s-1)$ can be cancelled); one has $\hat{y} = s^2\hat{\xi}$ hence

$$\hat{y} = \frac{s(s+1)}{(s+2)^2}\hat{u} \tag{12}$$

Similarly, $\hat{z} = 0$ where z is the variable defined in Example 2.

The following facts are standard (MacLane and Birkhoff 1968). Let $w_1, \ldots, w_q$ be elements of Λ; these elements are $\mathbf{R}$-linearly independent iff $\hat{w}_1, \ldots, \hat{w}_q$ are $\mathbf{F}$-linearly

independent (in particular, an element λ of Λ is torsion iff $\hat{\lambda} = 0$). One has $rk(\Lambda) = \dim(\hat{\Lambda})$ (i.e. the dimension of $\hat{\Lambda}$ over $\mathbf{F}$). Take two systems Λ_1 and Λ_2 such that $\Lambda_1 \subset \Lambda_2$; then $\hat{\Lambda}_1$ coincides with $\hat{\Lambda}_2$ iff the quotient module Λ_2/Λ_1 is torsion.

2.3. *Dynamics*

2.3.1. *Definition and examples.* A (linear) dynamics $\mathcal{D}$ is a system in which we distinguish a finite set $\boldsymbol{u} = \{u_1, \ldots, u_m\}$ of input variables such that the quotient module $\mathcal{D}/[u]$ is torsion (Fliess 1990, 1992 a). The torsion of $\mathcal{D}/[u]$ means that any element in $\mathcal{D}$ can be calculated from $\boldsymbol{u}$ by a linear differential equation.

Example 1 (continued): An equation of $\mathcal{D}/[u]$ is

$$D(s)\bar{\xi} = 0 \tag{13}$$

where $\bar{\xi}$ is the column matrix whose components are the images of the elements ξ_i by the canonical epimorphism $\mathcal{D} \to \mathcal{D}/[u]$ Notice that $\mathcal{D}/[u] = [\bar{\xi}]$ D is full rank over $\mathbf{F}$ (i.e. the normal rank of $D(s)$ is 4), so that the module defined by (13) is torsion.

Example 2 (continued): An equation of $\mathcal{D}/[u]$ is again (13), i.e. $s^2(s-1)(s+2)^2\bar{\xi} = 0$. Hence, $\mathcal{D}/[u] = [\bar{\xi}]$ is torsion.

The input $\boldsymbol{u}$ is said to be independent iff the module $[u]$ is free of rank m. We also distinguish a finite set $\boldsymbol{y} = \{y_1, \ldots, y_p\}$, called the output (by an equation such as (2)). See (Fliess 1990) for the connection with state-variable representation (which is, to some extent, a particular case of polynomial matrix descriptions considered below). In everything that follows, $\boldsymbol{u}$ is assumed to be independent.

2.3.2. *Connection with PMDs.* Let us establish the connection with Rosenbrock's polynomial matrix descriptions in the general case.

A partial state (sometimes called a pseudo state) of $\mathcal{D}$ is a subset $\{\xi_1, \ldots, \xi_r\}$ of $\mathcal{D}$ such that $\mathcal{D} = [\xi, u]$ Let us construct the module $\mathcal{D}$ as the module M above; $\mathcal{F}$ is a free module with basis

$$\mathbf{w} = \{\xi_1, \ldots, \xi_r, \ldots, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$$

thus $rk(\mathcal{F}) = r + m := k$. As $\mathcal{D}[u]$ is torsion, $rk(\mathcal{D}) = rk[u] = m$ (from above). Therefore, by (10) $rk(\Phi) = k - n = m$, i.e. $k = n + m$. As a result, $r = n$, i.e. the number of components of the partial state is equal to the number of independent equations.

Consider the equation (11) of $\mathcal{D}$ with the appropriate variables, i.e. $\boldsymbol{w} = \{\xi_1, \ldots, \xi_n, u_1, \ldots, u_m\}$. It can be written in the form (1), where $D(s) \in \mathbf{R}^{n\times n}$ and $N(s) \in \mathbf{R}^{n\times m}$.

An equation of $\mathcal{D}/[u]$ is (13). Hence, as $\mathcal{D}/[u]$ is torsion, D is full rank over $\mathbf{F}$, i.e. invertible.

As $y_i \in \mathcal{D}, 1 \le i \le p$, there exist polynomial matrices $Q(s) \in \mathbf{R}^{p\times n}$ and $W(s) \in \mathbf{R}^{p\times m}$ such that (2) holds.

Remark 1 Strict equivalence: The partial state ξ and even the number n of its components are not uniquely defined from $\mathcal{D}$. Moreover, the matrices D, N, Q and W are not uniquely defined once the partial state has been chosen. As a matter of fact, (1) is clearly equivalent to $UD\xi = UNu$, where U is any unimodular polynomial matrix. Now, consider the dynamics $y = \xi = u$. Equation (2) can be written in this case $y = 1\xi + 0u$, or $y = 0\xi + 1u$. □

$$\begin{tikzcd} {[\xi, u]} \arrow[rr, "\mu"] & & {[\xi', u]} \\ & {[u, y]} \arrow[ul, "i"] \arrow[ur, "i'"'] & \end{tikzcd}$$

Figure 1. Strict equivalence of PMDs.

More generally, consider two PMDs, with the same input $\boldsymbol{u}$ and the same output $\boldsymbol{y}$, and associated with two partial states ξ and ξ'. These PMDs are strictly equivalent (Rosenbrock 1970, Fuhrmann 1977) iff $[\xi, u] \simeq [\xi', u]$ (Perbeno 1977, Rudolph 1994), or more specifically iff the diagram in Fig. 1 is commutative (in this diagram, i and i' are canonical injections and μ is an isomorphism).

2.3.3. *Transfer matrix.* As $\mathcal{D}/[u]$ is torsion, $\hat{\mathcal{D}} = [\hat{u}]$ Therefore, there exists a rational matrix $G \in \mathbf{F}^{p\times m}$ such that $\hat{y} = G\hat{u}$. This matrix G is unique, because $\{\hat{u}_1, \ldots, \hat{u}_m\}$ is a basis of $\hat{\mathcal{D}}$ (from above), and G is the transfer matrix of $\mathcal{D}$ (Fliess 1994).

In the case where $\mathcal{D}$ is defined by (1) and (2), one obtains of course

$$G = QD^{-1}N + W \tag{14}$$

Example 1 (continued)**:** One has $G(s) = 1/(s+2)$.

Example 2 (continued)**:**

$$G(s) = \frac{s(s+1)}{(s+2)^2}$$

2.3.4. *Controllability.* A system Λ or a dynamics $\mathcal{D}$ is said to be controllable iff it is a free **R**-module (Fliess 1990).

This definition of controllability was proven by Fliess (1992) to be equivalent to the Willems controllability (Willems 1991). The following proposition proves that it is also equivalent to Rosenbrock's controllability in the case of a system defined by a PMD. A similar result concerning observability is obtained in Proposition 2 below.

Proposition 1: *Assume that the dynamics $\mathcal{D}$ is given by the PMD* (1) *and* (2). *Then $\mathcal{D}$ is controllable iff D and N are left-coprime.*

Proof: Let L be the greatest common left divisor of D and N, i.e. $D = LD^0, N = LN^0$, where D^0 and N^0 are left-coprime. Equation (1) is equivalent to

$$\left.\begin{aligned} D^0(s)\xi - N^0(s)u &= z \\ L(s)z &= 0 \end{aligned}\right\} \tag{15}$$

The latter equation is the equation of the torsion submodule T of $\mathcal{D}$ (i.e. $T \simeq [z]$). Therefore, T is equal to 0 iff L is unimodular, i.e. iff D and N are left-coprime. □

Remark 2: By definition, a dynamics $\mathcal{D}$ is controllable iff the module $\mathcal{D}$ is free, i.e. has a basis $\{\xi_1, \ldots, \xi_m\}$. Hence, such dynamics can be represented by a right-form

$y = N_R(s)\xi, u = D_R(s)\xi$, where $D_R(s)$ is square and invertible over $\mathbf{F}$ (Fliess 1994). Conversely, such a right-form is controllable (Kailath 1980). □

Remark 3: In the case where $\mathcal{D}$ is not controllable, let $\bar{\bar{\xi}}, \bar{\bar{u}}$ and $\bar{\bar{y}}$ denote the column matrices whose elements are the images of the elements of ξ, u and y by the canonical epimorphism $\mathcal{D} \to \mathcal{D}/T$. The equation of $\mathcal{D}/T$ is

$$\left.\begin{aligned} D^0(s)\bar{\bar{\xi}} &= N^0(s)\bar{\bar{u}} \\ \bar{\bar{y}} &= Q(s)\bar{\bar{\xi}} + W(s)\bar{\bar{u}} \end{aligned}\right\} \tag{16}$$

□

Definition 3: The free module $\mathcal{D}/T \simeq \Phi$, viewed as a dynamics with input $\bar{\bar{u}}$ and output $\bar{\bar{y}}$, is called the controllable dynamics of $\mathcal{D}$. □

Remark 4: As $\mathcal{D}/T \simeq \Phi$, by a slight abuse of notation, the projection $\mathcal{D} \to \mathcal{D}/T$ can be identified with the projection $\mathcal{D} \to \Phi$; in addition, as $[u]$ is free, it is identified with a submodule of Φ (isomorphic to $[\bar{\bar{u}}]$ and still denoted by $[u]$). By the former identification, the quotient modules $[\bar{\bar{\xi}}]$ and $[\bar{\bar{y}}]$ are identified with submodules of Φ denoted by $\Phi \cap [\xi]$ and $\Phi \cap [y]$ respectively. □

Example 1 (continued): Denote by $\mathcal{D}$ the dynamics defined in this example. One can choose

$$L(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, $T \simeq [z_3]$ where $sz_3 = 0$; T is non-zero, therefore $\mathcal{D}$ is uncontrollable. The controllable dynamics Φ is defined by (16) with

$$D^0(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s(s+1) & s+2 \\ 0 & 0 & 0 & s+2 \end{bmatrix}, \quad N^0(s) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Example 2 (continued): $T \simeq [z]$ where $z = s(s+2)^2\xi - (s+1)u$ satisfies $s(s-1)z = 0$. $\mathcal{D}$ is uncontrollable and the controllable dynamics Φ is defined by $s(s+2)^2\bar{\bar{\xi}} = (s+1)u$.

2.3.5. *Observability.* The dynamics $\mathcal{D}$ is said to be observable iff $\mathcal{D} = [y, u]$ (Fliess 1990). In other words, $\mathcal{D}$ is observable iff every element of $\mathcal{D}$ can be expressed as a linear combination of the components of y and u and of their derivatives of any order.

Definition 4: $[y, u]$ viewed as a dynamics with input u and output y, is called the observable dynamics of $\mathcal{D}$. □

Note that controllability and observability were proven to be dual notions in the module-theoretic setting of this paper (Rudolph 1996).

Proposition 2: *Assume that the dynamics $\mathcal{D}$ is given by the PMD* (1) *and* (2). *Then, $\mathcal{D}$ is observable iff D and Q are right-coprime.*

Proof: Let R be a greatest common right divisor of D and Q, i.e. $D = {}^0D\,R, Q = {}^0Q\,R$, where ${}^0D$ and ${}^0Q$ are right-coprime. Equations (1) and (2) are equivalent to

$$ {}^0D(s)\xi = N(s)u, \quad y = {}^0Q(s)\xi + W(s)u \tag{17}$$

$$ R(s)\xi = \xi \tag{18}$$

Let $\tilde{\xi}$ and $\tilde{\xi}$, respectively, denote the column matrices whose elements are the images of the elements of ξ and ξ by the canonical epimorphism $\mathcal{D} \to \mathcal{D}/[y,u]$ One obtains

$$ M(s)\tilde{\xi} = 0, \quad R(s)\tilde{\xi} = \tilde{\xi} \tag{19}$$

where $M = [{}^0D^{\mathrm{T}}\ {}^0Q^{\mathrm{T}}]^{\mathrm{T}}$. As ${}^0D$ and ${}^0Q$ are right-coprime, M is left-invertible, so that (19) implies $\tilde{\xi} = 0$. As a result, an equation of $\mathcal{D}/[y,u]$ is

$$ R(s)\tilde{\xi} = 0 \tag{20}$$

Now, $\mathcal{D} = [y,u]$ iff $\mathcal{D}/[y,u] = 0$, i.e. iff R is unimodular. □

Remark 5: According to the definition of a partial state (see § 2.3.2), dynamics $\mathcal{D}$ with output y is observable iff the output y can be chosen as a partial state of $\mathcal{D}$, i.e. iff $\mathcal{D}$ can be represented by a left-form $D_L(s)y = N_L(s)u$, where $D_L(s)$ is square and invertible over **F** (this remark is the dual of Remark 2). □

Example 1 (continued): One can choose

$$ R(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s^2(s+1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

and an equation of $\mathcal{D}/[y,u]$ is $s^2(s+1)\tilde{\xi}_3 = 0$. This module is non-zero, hence $\mathcal{D}$ is unobservable. The observable dynamics $[y,u]$ is defined by (17) with

$$ {}^0D(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s(s+2) \\ 0 & 0 & 0 & s+2 \end{bmatrix}, \quad {}^0Q(s) = Q(s) $$

Example 2 (continued): One has $\xi = s^2\xi$ (i.e. $R(s) = s^2$). $\mathcal{D}$ is unobservable. The observable dynamics is defined by $(s-1)(s+2)^2\xi = s(s-1)(s+2)u, y = \xi$.

2.3.6. *Transfer dynamics*. The module $\Phi \cap [y,u]$ (which is unique up to isomorphism, and which is defined using the identification of Remark 4), is the observable dynamics of the controllable dynamics of $\mathcal{D}$; it is also a free submodule of $[y,u]$ i.e. the controllable dynamics of the observable dynamics of $\mathcal{D}$. It is called the transfer module, or the transfer dynamics, of $\mathcal{D}$ (Rudolf 1994).

From Remark 5 above, the output y is a partial state of $[y,u]$ so that the transfer dynamics can be described by an equation

$$ D_0(s)\bar{\bar{y}} = N_0(s)u \tag{21}$$

(with the notation from Remark 4). The matrices D_0 and N_0 are left-coprime, and the transfer matrix of $\mathcal{D}$ is then equal to the irreducible fraction

$$G = D_0^{-1} N_0 \tag{22}$$

Example 2 (continued): The observable dynamics $[y, u]$ can be written

$$(s-1)(s+2)^2 y = s(s-1)(s+1)u$$

i.e. $(s-1)z' = 0$ and $(s+2)^2 y = s(s+1)u + z'$. The equation $(s-1)z' = 0$ defines the torsion submodule of $[y, u]$ i.e. $T \cap [y, u]$ An equation of the transfer dynamics is (21) with $D_0(s) = (s+2)^2$ and $N_0(s) = s(s+1)$.

3. Poles and zeros

3.1. *System poles*

Definition 5: The system poles of a dynamics $\mathcal{D}$ are the Smith zeros of the module $\mathcal{D}/[u]$ which is called the module of system poles. □

This module, as well as the other modules of poles and zeros defined in the sequel, are not the same as those introduced by Wyman and Sain (1981 a) (and used by several other authors).

If $\mathcal{D}$ is given by the PMD (1) and (2) its system poles are the Smith zeros of $D(s)$.

Example 1 (continued): The module of system poles is defined by (13). The Smith zeros of $D(s)$ are $\{-2, -1, 0, 0\}$, which is the set of system poles. The order of the system pole at $s = 0$ is equal to 2.

Example 2 (continued): The module of system poles is defined by $s^2(s-1)(s+2)^2\xi = 0$ and the set of system poles is $\{-2, -2, 0, 0, 1\}$. The orders of the system poles at $s = -2$ and $s = 0$ are equal to 2.

Definition 6: $\mathcal{D}$ is said to be internally stable iff none of its system poles lies in the closed right half-plane. □

3.2. *Controllable poles and observable poles*

Definition 7: The controllable poles (respectively, the observable poles) of $\mathcal{D}$ are the system poles of its controllable dynamics (respectively, of its observable dynamics). In other words, the controllable poles are the Smith zeros of the module $\Phi/[u]$ (using the identification of Remark 4) and the observable poles are the Smith zeros of the module $[y, u]/[u]$ Those modules are called the module of controllable poles and the module of observable poles respectively.

Example 1 (continued): The controllable poles are the Smith zeros of ${}^0D(s)$, i.e. $\{0, -1, -2\}$. The observable poles are the Smith zeros of ${}^0D(s)$, i.e. $\{-2\}$.

Example 2 (continued): The set of controllable poles is $\{-2, -2, 0\}$ (where the order of the controllable pole at $s = -2$ is 2), and the set of observable poles is $\{-2, -2, 1\}$ (where the order of the observable pole at $s = -2$ is 2).

Definition 8: $\mathcal{D}$ is said to be input–output stable iff none of its observable poles lies in the closed right half-plane. □

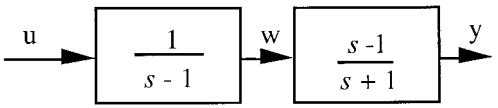

Figure 2. System (a).

In the case of single-input–single-output systems, this is equivalent to saying that the semi-cancellable fraction associated with $\mathcal{D}$ is stable (Bourlès 1994). Roughly speaking, in our terminology a system is input–output stable iff its external behaviour (i.e. the one involving only u and y) is stable, whatever the initial conditions are: only internal and hidden variables can have an unstable behaviour: see Example 3 below. See also Definition 10 below for the comparison with the usual terminology found in the literature.

Example 3: Consider, as in Bourlès (1994) and Fliess and Bourlès (1996), the system (a) (or more precisely, dynamics (a), according to the terminology of § 2.3.1), described by the block diagram in Fig. 2.

The equations of system (a) are assumed to be $u = (s - 1)w = (s + 1)y$. Hence, the observable dynamics is defined by $(s + 1)y = u$, and the system (a) is input–output stable. Now, the internal variable w satisfies $(s - 1)w = u$, hence 1 is a system pole. As a result, the system (a) is not internally stable. Note that the system pole 1 is an output-decoupling zero (see § 3.4.2).

3.3. *Transmission poles*

Definition 9: The transmission poles of $\mathcal{D}$ are the system poles of its transfer dynamics, i.e. the Smith zeros of the module $(\Phi \cap [y, u])/[u]$ which is called the module of transmission poles. □

Hence, the transmission poles are equal to the Smith zeros of the matrix D_0 defined by (21). Let

$$G^{\bullet} = \begin{bmatrix} \operatorname{diag}\{\varepsilon_i/\Psi_i\} & 0 \\ 0 & 0 \end{bmatrix} \tag{23}$$

be the Smith–MacMillan form of the transfer matrix G. Let ρ be the rank of G over $\mathbf{F}$. By (22), the $\varepsilon_i, 1 \le i \le \rho$, are the invariant factors of N_0, whereas the $\Psi_i, 1 \le i \le \rho$, are the ρ last invariant factors of D_0 in reverse order; the other invariant factors of D_0 are equal to 1 (Rosenbrock 1970, Kailath 1980). As a result, the transmission poles of $\mathcal{D}$ are the roots over $\mathbb{C}$ of the polynomials $\Psi_i(s), 1 \le i \le \rho$, including multiplicities (these roots are called the poles, or the Smith–MacMillan poles, of G (Rosenbrock 1970).

Definition 10: $\mathcal{D}$ is said to be transfer stable iff none of its transmission poles lies in the closed right half-plane. Note that this type of stability is the one usually considered in input–output approaches (Bourlès and Colledani 1995). □

Example 2 (continued): $\mathcal{D}$ is neither internally nor input–output stable, because 1 is an observable pole. However, the transfer function is $G(s) = s(s + 1)/(s + 2)^2$, hence the set of transmission poles is $\{-2, -2\}$ and $\mathcal{D}$ is transfer stable. Consider the

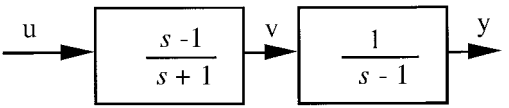

Figure 3. System (*b*).

solutions $y(.), u(.)$ and $z'(.)$. If u is such that $\lim_{t\to\infty} u(t) = 0$, one does not have $\lim_{t\to\infty} y(t) = 0$, except if $y(0) = u(0) = z'(0) = 0$.

Example 4: Take now the reverse block diagram of that in Fig. 2 (Kailath 1980, Bourlès 1994, Fliess and Bourlès 1996). One then obtains the system (*b*), described by the block diagram in Fig. 3.

It is assumed that $(s-1)u = (s+1)v, (s-1)y = v$, hence $(s^2-1)y = (s-1)u$. The transfer function of the system (*b*) is the same as that of system (*a*), i.e.

$$\frac{s-1}{s^2-1} = \frac{1}{s+1}$$

The only transmission pole is -1, hence the system (*b*) is transfer stable. However, the observable poles are -1 and 1. Consequently, the system (*b*) is not input–output stable according to Definition 8 (Bourlès 1994). Note that the system pole 1 is an input-decoupling zero (see § 3.4.1).

Remark 6: The systems (*a*) and (*b*) are two different series interconnections of subsystems. In the module theoretic approach used here, the interconnection of linear systems is defined as a fibred sum of modules (Fliess and Bourlès 1996). Examples 3 and 4 show that interconnecting controllable (respectively, observable) linear systems may give rise to an uncontrollable (respectively, unobservable) one. Callier and Nahum (1975) and Fuhrmann (1975) gave explicit necessary and sufficient conditions for such phenomena to occur when the systems under consideration are defined by PMDs; see also Anderson and Gevers (1981). These conditions can be obtained using the general definition of interconnection given by Fliess and Bourlès (1996). As a matter of fact, consider the series interconnection in Fig. 4 below, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are two dynamics such that the dimension p_1 of the output y_1 of $\mathcal{D}_1$ is equal to the dimension m_2 of the input u_2 of $\mathcal{D}_2$.

Let $\mathcal{D}$ be the dynamics resulting from this interconnection; $\mathcal{D}$ is the fibred sum written $\coprod_{y_1=u_2}(\mathcal{D}_1, \mathcal{D}_2)$ (Fliess and Bourlès 1996). This module is defined by the set of equations defining $\mathcal{D}_1$ and $\mathcal{D}_2$, plus the equation $y_1 = u_2$. Suppose that $\mathcal{D}_1$ is observable and is defined by

$$D_1(s)y_1 = N_1(s)u_1$$

and that $\mathcal{D}_2$ is controllable and is defined by

$$D_2(s)\xi_2 = u_2, \quad y_2 = Q_2(s)\xi_2$$

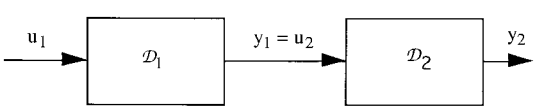

Figure 4. Series interconnection $\mathcal{D}$.

(see Remark 5 and Remark 2). By adding the equation $y_1 = u_2$, we obtain the equations of $\mathcal{D}$, i.e.

$$D_1(s)D_2(s)\xi = N_1(s)u_1, \quad y_2 = Q_2(s)\xi$$

By Proposition 1, $\mathcal{D}$ is controllable iff D_1D_2 and N_1 are left-coprime, and, by Proposition 2, $\mathcal{D}$ is observable iff D_1D_2 and Q_2 are right-coprime. These conditions are those given by Callier and Nahum (1975).

3.4. *Decoupling zeros (or hidden modes)*

3.4.1. *Input-decoupling zeros*

Definition 11: The input-decoupling-zeros (i.d.z.) of $\mathcal{D}$ are the Smith zeros of the module $\mathcal{D}$ (Fliess 1991), which is called the module of input-decoupling zeros. □

Obviously, $\mathcal{D}$ is controllable iff it has no i.d.z.

If $\mathcal{D}$ is defined by the PMD (1) and (2), its i.d.z. are the Smith zeros of L (see the proof of Proposition 1) or (what is equivalent) the Smith zeros of $[D \;\; N]$ (note that this is the definition given by Rosenbrock (1970)). If, more generally, $\mathcal{D}$ is defined by an equation of the form (11), its i.d.z. are the Smith zeros of S (or S^{T}). As for controllability, they are independent of the input.

Definition 12: $\mathcal{D}$ is said to be stabilizable iff none of its i.d.z. lies in the closed right half-plane (hence, stabilizability is a notion which is independent of the input). □

Example 1 (continued)**:** The set of input-decoupling zeros is $\{0\}$ and $\mathcal{D}$ is not stabilizable.

Example 2 (continued)**:** The set of input-decoupling zeros is $\{0, 1\}$: $\mathcal{D}$ is not stabilizable.

3.4.2. *Output-decoupling zeros*

Definition 13: The output-decoupling-zeros (o.d.z.) of $\mathcal{D}$ are the Smith zeros of the module $\mathcal{D}/[y, u]$ (Fliess 1991), which is called the module of output-decoupling zeros. □

Clearly, the dynamics $\mathcal{D}$ is observable iff it has no o.d.z.

If $\mathcal{D}$ is defined by the PMD (1) and (2), its o.d.z. are the Smith zeros of R (see the proof of Proposition 2) or (what is equivalent) the Smith zeros of $[D^{\mathrm{T}} \;\; Q^{\mathrm{T}}]^{\mathrm{T}}$ (this is the definition given by Rosenbrock 1970).

Definition 14: $\mathcal{D}$ is said to be detectable iff none of its o.d.z. lies in the closed right half-plane. □

Example 1 (continued)**:** The set of o.d.z. is $\{0, 0, -1\}$ (where the order of the o.d.z. at $s = 0$ is 2) and $\mathcal{D}$ is not detectable.

Example 2 (continued)**:** The set of o.d.z. is $\{0, 0\}$ (where the order of the o.d.z. at $s = 0$ is 2); $\mathcal{D}$ is not detectable.

3.4.3. *Input-output decoupling zeros*

Definition 15: The input–output-decoupling zeros (i.o.d.z.) are the Smith zeros of the module $T/(T \cap [y, u])$, which is called the module of input–output decoupling zeros. □

Remark 7: $T/(T \cap [y,u])$ can be viewed as the part of the dynamics which is uncontrollable and unobservable. Note that the Smith zeros of $T \cap [y,u]$ are the i.d.z. of the observable dynamics and that the Smith zeros of $\Phi/(\Phi \cap [y,u])$ are the o.d.z. of the controllable dynamics. The algorithm mentioned in § 1 to calculate the i.o.d.z. is clarified by the following result. □

Proposition 3: The following equalities hold:

$$\{\text{i.o.d.z}\} = \begin{cases} = \{\text{i.d.z.}\} - \{\text{i.d.z. of the observable dynamics}\} & (24) \\ = \{\text{o.d.z.}\} - \{\text{o.d.z. of the controllable dynamics}\} & (25) \end{cases}$$

Proof: (24) is a direct consequence of Definition 15 of the Lemma. In addition, one has

$$\frac{\mathcal{D}}{[y,u]} \simeq \frac{T}{T \cap [y,u]} \oplus \frac{\Phi}{\Phi \cap [y,u]}$$

and (25) is now a consequence of the lemma. □

Remark 8: Assuming that the dynamics are defined by a PMD (1) and (2), there are two equivalent ways to calculate the i.o.d.z. Using (24), one has first to calculate the Smith zeros of $[D(s)\ N(s)]$ (i.e. the i.d.z.), then to calculate the Smith zeros of $[{}^0D(s)\ N(s)]$ (i.e. the i.d.z. of the observable dynamics) and finally to subtract those two sets. Using (25) one has first to calculate the Smith zeros of $[D^{\mathrm{T}}(s)\ Q^{\mathrm{T}}(s)]^{\mathrm{T}}$ (i.e. the o.d.z.), then to calculate the Smith zeros of $[D^{0\mathrm{T}}(s)\ Q^{\mathrm{T}}(s)]^{\mathrm{T}}$ (i.e. the o.d.z. of the controllable dynamics) and finally to subtract those two sets. Note that (24) and (25) allow us to calculate the degree of an i.o.d.z., but not its structural indices, as opposed to Definition 15. Let us now detail how the latter can be calculated. □

Consider (17) and (18); let ${}^0L(s)$ be a greatest common left divisor of ${}^0D(s)$ and $N(s)$, and set ${}^0D(s) = {}^0L(s)\,{}^0D^0(s)$, $N(s) = {}^0L(s)\,{}^0N(s)$. Set $z' = {}^0D^0(s)\xi - {}^0N(s)u$. The module $[z']$ is the torsion submodule of $[y,u]$ i.e. $[z'] = T \cap [y,u]$ An equation of this module is ${}^0L(s)z' = 0$. An equation of T is (15), i.e. $L(s)z = 0$, and $[z'] \subset [z]$ hence there exists a matrix $\tilde{L}(s)$ (square and invertible over $\mathbf{F}$) such that $z' = \tilde{L}(s)z$; it follows that $L(s) = {}^0L(s)\tilde{L}(s)$. As ${}^0L(s)$ is invertible over $\mathbf{F}$, this equation completely defines $\tilde{L}(s)$.

Let $\hat{z}$ and $\hat{z}'$ be the column matrices whose components are the images of those of z and z' by the canonical epimorphism $T \to T/(T \cap [y,u])$. One has $\hat{z}' = 0$, hence an equation of $T/(T \cap [y,u])$ is $\tilde{L}(s)\hat{z} = 0$. In other words, the i.o.d.z. are the Smith zeros of $\tilde{L}(s)$.

Example 1 (continued)**:** The matrices ${}^0D(s)$ and $N(s)$ are right-coprime, hence the torsion submodule of $[y,u]$ i.e. $T \cap [y,u]$ is 0. Therefore, $T/(T \cap [y,u]) \simeq T$ and the set of i.o.d.z. is equal to the set of i.d.z., i.e. $\{0\}$.

Example 2 (continued)**:** Using the result of Remark 8, the set of i.o.d.z. is found to be $\{0\}$.

3.4.4. *Hidden modes*

Definition 16: The hidden modes are the Smith zeros of the module $\mathcal{D}/(\Phi \cap [y,u])$, which is called the module of hidden modes. □

Proposition 4: *One has*

$$\{hidden\ modes\} = \{i.d.z.\} + \{o.d.z.\} - \{i.o.d.z.\} \tag{26}$$

Proof: Using the lemma, one obtains $\mathcal{D}/(\Phi \cap [y,u]) \simeq T \oplus \Phi/(\Phi \cap [y,u])$, hence $\{\text{hidden modes}\} = \{\text{i.d.z.}\} + \{\text{o.d.z. of the controllable dynamics}\}$, and (26) is obvious from (25). □

Example 1 (continued)**:**

$$\{\text{hidden modes}\} = \{0\} + \{0,0,-1\} - \{0\} = \{0,0,-1\}$$

Example 2 (continued)**:**

$$\{\text{hidden modes}\} = \{0,1\} + \{0,0\} - \{0\} = \{0,0,1\}$$

3.5. *Invariant zeros*

Definition 17: The invariant zeros of $\mathcal{D}$ are the Smith zeros of the module $\mathcal{D}/[y]$ which is called the module of invariant zeros.

Assume that $\mathcal{D}$ is given by the PMD (1) and (2), and let $\check{\xi}$ and $\check{u}$ denote the column matrices whose components are the canonical images of the elements of ξ and u by the canonical epimorphism $\mathcal{D} \to \mathcal{D}/[y]$ The equation of $\mathcal{D}/[y]$ is

$$P(s)\begin{bmatrix} \check{\xi} \\ \check{u} \end{bmatrix} = 0$$

where P is the system matrix defined by

$$P = \begin{bmatrix} D & -N \\ Q & W \end{bmatrix} \tag{27}$$

Hence, the invariant zeros of $\mathcal{D}$ are the Smith zeros of $P(s)$.

Example 1 (continued)**:** As $P(s)$ is square and as its normal rank is full, the invariant zeros are the roots of its determinant. Hence, their set is $\{0,0,-1\}$. It can be verified that the order of the invariant zero at s= 0 is 2.

Example 2 (continued)**:** One can calculate the invariant zeros using the same rationale as above. More directly, $\mathcal{D}/[y]$ is defined by the equations $s^2\check{\xi} = 0, s(s-1)(s+1)\check{u} = 0$. This module is torsion, and one has

$$\mathcal{D}/[y] \cong \frac{\mathbf{R}}{s^2\mathbf{R}} \oplus \frac{\mathbf{R}}{s(s-1)(s+1)\mathbf{R}}$$

so that the set of invariant zeros is $\{-1,0,0,0,1\}$. Note that the structural indices of $\mathcal{D}/[y]$ at 0 are 1 and 2, hence the order of the invariant zero at $s = 0$ is 2 (whereas its algebraic multiplicity is 3 and its geometric multiplicity is 2).

Remark 9: The modules $[y,u]/[y]$ and $\Phi/(\Phi \cap [y])$ can also be considered. They are the modules of invariant zeros of the observable dynamics and of the controllable dynamics respectively. □

3.6. *Transmission zeros*

Definition 18: The transmission zeros of $\mathcal{D}$ are the invariant zeros of its transfer dynamics, i.e. the Smith zeros of $(\Phi \cap [y,u])/(\Phi \cap [y])$. This module is called the module of transmission zeros. □

Consider the equation (21) of $\Phi \cap [y,u]$ The equation of $(\Phi \cap [y,u])/(\Phi \cap [y])$ is

$$N_0(s)\mathring{u} = 0 \tag{28}$$

where $\mathring{u}$ is the column matrix whose elements are the images of the elements of u by the canonical epimorphism $\Phi \cap [y,u] \to (\Phi \cap [y,u])/(\Phi \cap [y])$. As a result, the transmission zeros are the Smith zeros of $N_0(s)$.

Now, consider the Smith–MacMillan form (23) of the transfer matrix G; the Smith zeros of $N_0(s)$ (i.e. the transmission zeros) are the roots over $\mathbb{C}$ of the polynomials $\varepsilon_i(s), 1 \leq i \leq \rho$, including multiplicities. These roots are called the zeros, or the Smith–MacMillan zeros, of G (Rosenbrock 1970, Morse 1973, Morse and Silverman 1974).

Example 1 (continued): $\mathcal{D}$ has no transmission zeros.

Example 2 (continued): The set of transmission zeros is $\{-1, 0\}$.

Definition 19: $\mathcal{D}$ is said to be minimum phase iff none of its transmission zeros lies in the closed right half-plane. □

3.7. *Blocking zeros*

Definition 20: The blocking zeros of $\mathcal{D}$ are the roots over $\mathbb{C}$ of $\varepsilon_1(s)$, including multiplicities. More specifically, $z \in \mathbb{C}$ is a blocking zero of $\mathcal{D}$ with order ω iff it is a root of $\varepsilon_1(s)$ with multiplicity ω. □

Note that ε is the greatest common divisor (GCD) of the minors of order ρ of N_0, whereas ε_1 is the GCD of the elements of N_0. Hence, a blocking zero β is a complex number such that $G(\beta) = 0$. Obviously

$$\{\text{blocking zeros}\} \subset \{\text{transmission zeros}\} \tag{29}$$

In the case of single-input–single-output systems, the above inclusion is an equality.

3.8. *System zeros*

The system zeros were defined by Rosenbrock (1974):

$$\{\text{system zeros}\} = \{\text{transmission zeros}\} + \{\text{hidden modes}\} \tag{30}$$

As shown by Rosenbrock (1974), in the case of a dynamics defined by a PMD, the system zeros can be calculated from the system matrix $P(s)$. For every matrix M of dimension $n \times m$ let $M^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$ denote the minor formed by the rows $\{i_1,\ldots,i_k\}$ and the columns $\{j_1,\ldots,j_k\}$ of M, assuming that $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq m$. One has

$$\{\text{roots of GCD}[P^{1,2,\ldots,n,i_1,\ldots,i_k}_{1,2,\ldots,n,j_1,\ldots,j_k}(s)]\} = \{\text{system zeros}\}$$

3.9. *Dynamical interpretation of poles and zeros and of their order*

Consider the behaviour associated with dynamics $\mathcal{D}$ (see § 1), and let σ be a pole or a zero with order ω. This means that $\mathbb{C}$-linear combinations of functions $t \to e^{\sigma t}, te^{\sigma t}, \ldots, t^{\omega-1}e^{\sigma t}$ are involved as follows.

(*a*) If σ is a system pole (respectively, and observable pole), then for $u = 0$, such combinations are components of the internal variables (respectively, the output variables).

(*b*) If σ is an i.d.z. (respectively, an o.d.z.; respectively, an i.o.d.z.), then such combinations are components of the internal variables and cannot be eliminated using the control (respectively,cannot be observed using u and y; respectively, can neither be eliminated using the control nor observed using u and y).

(*c*) If σ is a transmission zero (respectively, a blocking zero), then some inputs (respectively, all inputs) which are expressed as such combinations are completely blocked, i.e. have no effect on the output (Desoer and Schulman 1974, Ferreira and Bhattacharyya 1977).

(*d*) If σ is an invariant zero, then some inputs that are expressed as such combinations are completely blocked for some specific initial conditions; see MacFarlane and Karkanias (1976) for more details in the case $\omega = 1$ and when the system is defined by a state-space realization.

4. Relationships between various poles and zeros

The theorems below are well-illustrated by the examples considered above and Examples 5 and 6 below; the relations (35), (37), (39) and (40) are well known (see § 5) but are proved here in a very concise and original manner.

Theorem 1: *The following equalities hold:*

$$\{\text{system poles}\} = \begin{cases} = \{\text{i.d.z.}\} + \{\text{controllable poles}\} & (31) \\ = \{\text{o.d.z.}\} + \{\text{observable poles}\} & (32) \end{cases}$$

$$\{\text{observable poles}\} = \{\text{transmission poles}\} + \{\text{i.d.z.}\} - \{\text{i.o.d.z.}\} \quad (33)$$

$$\{\text{controllable poles}\} = \{\text{transmission poles}\} + \{\text{o.d.z.}\} - \{\text{i.o.d.z.}\} \quad (34)$$

$$\{\text{system poles}\} = \{\text{transmission poles}\} + \{\text{hidden modes}\} \quad (35)$$

Proof: The above lemma will be used throughout this proof (and throughout the proof of Theorem 2 below).

As $[u]$ is free, one has (with the identifications of Remark 4)

$$\mathcal{D}/[u] = (T \oplus \Phi)/[u] \simeq T \oplus \Phi/[u]$$

and this proves (31). In addition

$$\mathcal{D}/[y,u] \simeq (\mathcal{D}/[u])/([y,u]/[u])$$

and this proves (32). One has also

$$[y,u]/[u] \simeq (\Phi \cap [y,u])/[u] \oplus (T \cap [y,u])$$

so that

$$\{\text{observable poles}\} = \{\text{transmission poles}\} + \{\text{i.d.z. of the observable dynamics}\}$$

and (33) is proved using (24). One has

$$\Phi/(\Phi \cap [y,u]) \simeq \frac{\Phi/[u]}{(\Phi \cap [y,u])/[u]}$$

hence

$$\{\text{controllable poles}\} = \{\text{transmission poles}\} + \{\text{o.d.z. of the controllable dynamics}\}$$

and (34) is proved using (25). The equality (35) is the consequence of, say, (32) and (33). □

Theorem 2:

(*a*) *One has always*

$$\{\textit{transmission zeros}\} + \{\textit{i.o.d.z.}\} \subset \{\textit{invariant zeros}\} \tag{36}$$

$$\{\textit{invariant zeros}\} \subset \{\textit{system zeros}\} \tag{37}$$

(*b*) *Assume that* $\rho = p$, *i.e. G is right-invertible; then*

$$\{\textit{transmission zeros}\} + \{\textit{i.d.z.}\} \subset \{\textit{invariant zeros}\} \tag{38}$$

(*c*) *Assume that* $\rho = m$, *i.e. that G is left-invertible* (*and* $\mathcal{D}$ *itself is said to be left-invertible* (Fliess 1889)); *then*

$$\{\textit{transmission zeros}\} + \{\textit{o.d.z.}\} \subset \{\textit{invariant zeros}\} \tag{39}$$

(*d*) Finally, assume that G is square and invertible (*i.e.* $p = m = \rho$); *then*

$$\{\textit{transmission zeros}\} + \{\textit{hidden modes}\} = \{\textit{invariant zeros}\} \tag{40}$$

Proof:

(*a*) One has

$$\mathcal{D}/[y] = (T \oplus \Phi)/[y] \simeq T/(T \cap [y]) \oplus \Phi/(\Phi \cap [y]) \tag{41}$$

$$(\Phi \cap [y,u])/(\Phi \cap [y]) \subset \Phi/(\Phi \cap [y])$$

$$\{\text{Smith zeros of } T/(T \cap [y,u])\} \subset \{\text{Smith zeros of } T/(T \cap [y])\}$$

so that (36) holds.

Let $\mathcal{T}\{.\}$ denote the torsion submodule of the module in brackets. One has

$$\mathcal{D}/[y,u] \simeq \Phi/(\Phi \cap [y,u]) \oplus T/(T \cap [y,u])$$

$$\supset T_1 \oplus T/(T \cap [y,u]) \tag{42}$$

where $T_1 \simeq \mathcal{T}\{\Phi/(\Phi \cap [y,u])\}$, hence

$$T_1 \simeq \mathcal{T}\{\Phi/(\Phi \cap [y])\}/\mathcal{T}\{(\Phi \cap [y,u])/(\Phi \cap [y])\}$$

By (42)

$$\{\text{Smith zeros of } T_1\} \subset \{\text{o.d.z.}\} - \{\text{i.o.d.z.}\}$$

and by (41)

$$\{\text{invariant zeros}\} = \{\text{transmission zeros}\} + \{\text{Smith zeros of } T_1\}$$

$$+ \{\text{Smith zeros of } T/(T \cap [y])\}$$

One has also

$$\{\text{Smith zeros of } T/(T \cap [y])\} \subset \{\text{i.d.z.}\}$$

and (37) is proved.

(*b*) In this case, $\dim [\hat{y}] = p$, hence $[y]$ is free (§ 2.2). Therefore, $T \cap [y] = 0$, and (41) becomes

$$\mathcal{D}/[y] \simeq T \oplus \Phi/[y] \supset T \oplus \frac{\Phi \cap [y,u]}{[y]}$$

and (38) is proved.

(*c*) In this case, $\mathcal{D}/[y]$ is torsion. One has

$$\mathcal{D}/[y,u] \simeq (\mathcal{D}/[y])/([y,u]/[y])$$

$$[y,u]/[y] \simeq (T \cap [y,u])/(T \cap [y]) \oplus (\Phi \cap [y,u])/(\Phi \cap [y]) \tag{43}$$

so that (39) is proved.

(*d*) In this case, $[y]$ is free and $\mathcal{D}/[y]$ is torsion, hence (40) becomes

$$[y,u]/[y] \simeq (T \cap [y,u]) \oplus ((\Phi \cap [y,u])/[y])$$

and (40) is proved. □

Remark 10: In case (*b*) (respectively (*c*)), the set of o.d.z. (respectively, i.d.z.) is generally not included in the set of invariant zeros, as shown by examples 5 and 6 below. (Schrader and Sain (1989) claimed that in the general case, the invariant zeros contain all transmission zeros, all the o.d.z. and some of the i.d.z. that are not i.o.d.z. This example shows that this is not true (note that Rosenbrock (1977) obtained this result in case (*c*) of Theorem 2, where it is correct). Hence, the inclusion (36) seems to be the finest one can obtain in the general case, and, to the best of our knowledge, it is new.) In case (*c*), $\mathcal{D}/[y]$ is torsion, hence one can consider the left-inverse dynamics with input $\boldsymbol{y}$ (Fliess 1989). In case (*d*), $\mathcal{D}$ is (left- and right-) invertible, and the Smith zeros of $\mathcal{D}/[y]$ are the system poles of the inverse dynamics; then (40) is consistent with (35) and the system zeros are nothing but the invariant zeros. □

Example 5: The dynamics

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = x_2 + u_2, \quad y = x_1$$

has the o.d.z. $s = 1$ but has no invariant zeros.

Example 6: The dynamics

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_2, \quad y_1 = x_1, \quad y_2 = x_2$$

has the i.d.z. $s = 1$ but has no invariant zeros.

5. Choosing suitable input variables

It has been noted above that stabilizability is a property which is independent of the choice of input variables. To better understand the significance of this (perhaps surprising) fact, let us consider the problem of choosing suitable input variables in the case where one is faced with a linear system $\mathcal{D}$ defined by a set of differential equations such as (11), where those variables are not *a priori* distinguished (Willems 1991 and Fliess 1990, 1992 a noted that controllability is a notion which is independent of the choice of input variables). The module-theoretic setting of the

present paper is very convenient for systematically determining such input variables. The following conditions have to be satisfied:

(*a*) $\mathcal{D}$ is stabilizable

(*b*) the input $\boldsymbol{u} = \{u_1, \ldots, u_m\}$ is independent

(*c*) $\mathcal{D}/[u]$ is torsion, i.e. $rk\mathcal{D} = m$.

Set $u^j = \{u_1, \ldots, u_j\}, j \geq 1$, and $u^0 = \varnothing$. Clearly, $[u^m]$ is free iff

$$rk\mathcal{D}/[u^j] = rk\mathcal{D}/[u^{j-1}] - 1, \quad 1 \leq j \leq m \tag{44}$$

As a result, an m-step iterative procedure algorithm can be used to choose suitable inputs. It can be described as follows:

Step 0. Calculate the i.d.z. of $\mathcal{D}$ (§ 3.4.1) and verify that $\mathcal{D}$ is stabilizable. If not, stop (the problem has no solution). Calculate $m = rk\mathcal{D}$. Assuming that $\mathcal{D}$ is defined by (11), one has $m = k - n$ (i.e. the number of variables minus the number of independent equations).

Step j. Suppose that u^{j-1} has been chosen such that

$$rk\mathcal{D}/[u^{j-1}] = m - (j - 1)$$

Then choose u_j such that (44) is satisfied.

For $j = m$ the procedure stops.

In practice, this algorithm consists of extracting from the matrix $S^{\mathrm{T}}(s) \in \mathbf{R}^{n \times k}$ (which is assumed to be of rank n over $\mathbf{F}$, i.e. redundant equations have been removed) n $\mathbf{R}$-linearly independent columns forming a submatrix $D(s) \in \mathbf{R}^{n \times n}$ which is full rank. Indeed, if

$$u_1 = w_{\sigma(1)}, \ldots, u_j = w_{\sigma(j)}, \quad 1 \leq j \leq m$$

where σ is a permutation of $\{1, \ldots, k\}$, the matrix of definition of $\mathcal{D}/[u^j]$ is obtained from $S^{\mathrm{T}}(s)$ by removing the columns numbered $\sigma(1), \ldots, \sigma(j)$. By reordering the columns of $S^{\mathrm{T}}(s)$ (hence the variables), this matrix is put in the form $[D(s) - N(s)]$ and (11) is put in the form (1).

Example 7 (Bourlès and Marinescu 1996 b): Consider the electrical circuit in Fig. 5. If $C_1 = C_2 = 2C$, this configuration corresponds to the π-equivalent scheme of an electrical transmission line used for purposes of analysis involving interconnection with other elements of an electric network; X, R and C denote the total series inductance, the total series resistance and the total shunt capacitance, respectively

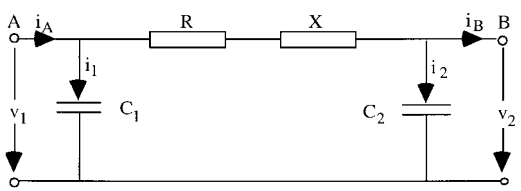

Figure 5. Circuit example.

(the shunt inductance is neglected). The equations that describe the circuit in Fig. 5 are

$$C_1\dot{v}_1 = i_1$$
$$C_2\dot{v}_2 = i_2$$
$$v_1 = R(i_A - i_1) + X\frac{\mathrm{d}}{\mathrm{d}t}(i_A - i_1) + v_2$$
$$i_A - i_1 = i_B + i_2$$

The variables of these equations are the voltages v_1 and v_2 and the currents i_1, i_2, i_A and i_B. The input, output and internal variables are not *a priori* distinguished. In this case, $m = 2$ and $\mathcal{D}$ is stabilizable (and even controllable, hence every non-zero element of $\mathcal{D}$ is free). Choose, for example, $u_1 = v_1$. Then by using the above algorithm it is easy to verify that neither i_1 not i_A can be the second input variable; one can choose $u_2 = i_2, i_B$ or v_2.

Remark 11: It is often necessary to choose $\boldsymbol{u}$ such that impulsive behaviours and derivatives of the input are avoided. This condition is satisfied iff no system pole at infinity is obtained (Verghese 1979, 1980). The above algorithm can be completed to systematically choose $\boldsymbol{u}$ such that the latter condition is also satisfied (see Bourlès and Marinescu 1996 b).

6. Concluding remarks

In this paper we have defined poles and zeros and we have studied their relationships in an intrinsic manner. In our opinion this viewpoint clarifies those notions, in that through it concise and precise definitions of them can be given; it also makes it possible to establish fundamental relatioships between various poles and zeros in a rather abstract but very concise manner. This approach is simple; only standard results about finitely generated modules over principal ideal domains have been utilized.

The relations (35), (37) and (40) were first established by Rosenbrock (1974) in the PMD framework, and (39) by Rosenbrock (1977); (38) can be considered as the dual of (39). To the best of our knowledge, the equalities (31)–(34) and the inclusion (36) (in the general case) are new. With our approach, the structural indices of an input–output decoupling zero can be calculated, and this point is also new (see Remark 8).

The notions of input–output stability and transfer stability have also been studied and detailed. By (33), a dynamics $\mathcal{D}$ is input–output stable iff it is transfer stable and all its i.d.z.s belonging to the closed right half-plane (if any) are i.o.d.z.s; in particular, the latter condition is satisfied if $\mathcal{D}$ is stabilizable. By (32), $\mathcal{D}$ is internally stable iff it is input–output stable and detectable; and by (35), $\mathcal{D}$ is internally stable iff it is transfer stable and none of its hidden modes lies in the closed right half-plane.

Stabilizability is a notion that is independent of control variables. The significance of this property has been clarified by considering the problem of systematically choosing suitable input variables of a linear system in the case where they are not *a priori* distinguished. Our module-theoretic setting is very convenient for solving this problem.

Discrete-time systems may be treated along the same lines with the formalism of (Fliess 1992 b).

As an extension of the module based framework of the present paper, poles and zeros at infinity of linear time-varying systems have been studied by Bourlès and Marinescu (1996 a).

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