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Periodic-polynomial interpretation for structural properties of linear periodic discrete-time systems

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Abstract

The structural properties of linear periodic discrete-time systems are analyzed in the periodic polynomial representation. It is shown that the classical polynomial approach for linear time-invariant systems can be extended to periodic systems. New definitions and properties are given in terms of skew polynomial rings and periodic difference algebra. Necessary and sufficient conditions for the characterization of reachability, controllability, observability and reconstructibility are given in this framework. © 1998 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Linear periodic discrete-time (LPDT) systems have been receiving a lot of attention because of the large variety of plants which can be modelled through such a description (see e.g., [19, 20] and related references).

Generally, as far as analysis of structural properties is concerned, the state space representation is well suited for linear time-varying (LTV) systems, since all of these properties can be defined in this framework [21, 24]. As a matter of fact, necessary and sufficient conditions for complete or h-step reachability and observability of LPDT systems have been characterized in this approach by a rank condition [17, 9, 10]. The use of linear time-invariant (LTI) associated systems, obtained by the technique of shift operator [19, 13] and cyclic transformation [23], are another efficient tool for such analysis. Modal observability and reachability have also been characterized using a Gramian matrix in [1]. A transfer matrix approach has been developed for the study of LPDT feedback of detectable and stabilizable systems [16, 8]; the transfer matrix under consideration is that of the LTI system obtained using the shift operator. However, as is well known, problems such as loss of reachability or observability cannot be taken into account in a transfer matrix formulation [15, 3].

The aim of this paper is to study various structural properties of LPDT systems in their periodic polynomial matrix description (PMD) [21], which includes the periodic state space framework as a special

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case. The particularity of such systems – as it will be shown below – is that they are defined over a non-commutative, non-integral and non-principal ideal ring $\mathfrak{R}$, the ring of periodic polynomials. Such a ring is very "exotic"; thus, before giving the characterizations of structural properties of LPDT systems, a preliminary study of some algebraic properties of $\mathfrak{R}$ is necessary.

The paper is organized as follows: in Section 2, some preliminaries and notations are presented. In Section 3, some mathematical results on algebraic properties of $\mathfrak{R}$ are given. In particular, a ring-isomorphism between $\mathfrak{R}$ and a ring $\mathscr{M}_{\mathscr{R}}$ of polynomial and structured matrices is established. As $\mathfrak{R}$ is not a Bezout ring, the notions of weak and strong coprimeness of a pair of polynomials in $\mathfrak{R}$ are introduced, and a necessary and sufficient condition is given for these properties to be equivalent. In Section 4, the above results are used to give a periodic PMD characterization of all structural properties of LPDT systems in terms of periodic PMDs. Section 5 is reserved for concluding remarks and Section 6 to appendices.

2. Notation and preliminaries

2.1. Notation

$:=$	equals by definition
$\mathscr{Z}$ (resp. $\mathscr{N}$)	set of integers (resp. non-negative integers)
$\mathbb{R}$	field of real numbers
S	set of real-valued sequences defined over $\mathscr{Z}$
0_S (resp. 1_S)	zero (resp. unit) element in S
$a(n)$	value of a sequence a belonging to S at time n
$a^{(k)}$	sequence belonging to S defined by $a^{(k)}(n)=a(n+k),\ k\in\mathscr{N}$
S_N	subset of S of N-periodic real-valued sequences ($a\in S_N \Leftrightarrow a^{(N)}=a$)
q	the usual forward-shift operator
δ	$:=q^N$
$\mathscr{A}[q]$	ring of all polynomials in q over $\mathscr{A}$ (where $\mathscr{A}$ is any ring)
$\mathscr{A}^{p\times m}$	set of $p\times m$ matrices ($p,m\in\mathscr{N}^*$) with entries belonging to the ring $\mathscr{A}$
$\mathscr{A}[q)$	ring consisting of quotients $p(q)/q^n,\ p(q)\in\mathscr{A}[q],\ n\in\mathscr{N}$
$\underline{\mathscr{M}}$	$:=\mathbb{R}[\delta]^{N\times N}$
$\underline{\mathscr{M}}_{\mathscr{R}}$	the subring of $\underline{\mathscr{M}}$ consisting of all polynomial matrices $M(\delta)$ such that $M(0)$ is upper triangular
M^{T}	transpose of a matrix M
$M_{i,j}$	(i,j) entry of the matrix M
$e(q)$	$:=[1\ q\ldots q^{N-1}]^{\mathrm{T}}$
$\mathrm{diag}(a_1,\ldots,a_N)$	diagonal matrix with $a_1,\ldots,a_N$ on the diagonal
I_N	identity matrix of dimension N
$S_N[[q^{-1}]]$	ring of all formal series of the form $f(q)=\sum_{i=0}^{\infty}q^{-i}f_i,\ i\in\mathscr{N}$ and $f_i\in S_N$

2.2. Preliminaries

2.2.1. Ring of periodic sequences S_N

With the usual elementwise addition and multiplication of sequences, S_N is a commutative and unit ring; a non-zero element a of S_N is invertible if and only if (iff) $a(n)\neq 0$, for every integer n, $0\leqslant n\leqslant N-1$; and a is non-zero divisor iff it is invertible; hence S_N is not an integral ring.

2.2.2. The skew polynomial ring $\mathfrak{R}$

The ring of N-periodic polynomials is $\mathfrak{R}:=S_N[q]$; every non-zero element $f(q)$ in $\mathfrak{R}$ is a polynomial of the form

$$f(q)=a_0+a_1q+\cdots+a_nq^n,\quad a_i\in S_N \text{ and } a_n\neq 0_s, \tag{2.1}$$

moreover, the "commutation" rule, between the indeterminate q and elements of S_N, is expressed by

$$q^k a_i = a_i^{(k)} q^k \tag{2.2}$$

for any $a_i \in S_N$ and $k \in \mathcal{N}$. The interpretation of this commutation rule is clear.

By the N-periodicity of the elements of S_N, one has $q^N a_i = a_i q^N$, i.e., $\delta a_i = a_i \delta$.

$\Re$ is called the ring of skew [2] polynomials in q over S_N determined by Eq. (2.2) [5].

2.2.3. Some particularities of $\Re$

$\Re$ is *not an integral* ring (as S_N is not so); therefore, it does not possess a quotient field, and many other properties of usual polynomial rings are lost. We will see below (a counterexample is given in Appendix A.2) that $\Re$ is *not a Bezout ring*; consequently, it is *not a principal ideal ring*. This means in particular that weak and strong coprimeness of polynomials over $\Re$ (here-below defined) do not coincide.

$\Re$ is *not an Euclidean* ring: for every non-zero polynomial $f(q)$ of the form (2.1), set $v(f) = n$ and $v(0) = -\infty$; v is only a valuation but not a degree function, because the following inequality can be strict:

$$v(f \cdot g) \leqslant v(f) \cdot v(g).$$

2.2.4. Coprimeness in $\Re$

Definition 2.1. (i) Two periodic polynomials are weakly left (resp. right) coprime in $\Re$ iff all their common left divisors (cld's) [resp. all their common right divisors (crd's)] in $\Re$ are unimodular. [3]

(ii) A pair $(a(q), b(q))$ of polynomials is strongly left (resp. right) coprime in $\Re$ iff $a(q)\Re + b(q)\Re = \Re$ (resp. $\Re a(q) + \Re b(q) = \Re$).

Strong coprimeness implies weak coprimeness, but the converse is true only over a Bezout ring [3].

2.2.5. Particular rings of periodic polynomial matrices

Set $\mathcal{M} := S_N[\delta]^{N\times N}$; the value of a matrix $F(\delta)$ at time $n, (n \in \mathcal{N})$, is denoted by $F(\delta, n)$. Obviously $F(\delta, n)$ is in $\underline{\mathcal{M}}$. The following proposition can be easily deduced from the fact that $\underline{\mathcal{M}}$ is a principal ideal ring [5].

Proposition 2.1. *$\mathcal{M}$ is a principal ideal ring* (*but it is not integral*).

Let $\mathcal{M}_{\mathcal{R}}$ be the set of $N \times N$ polynomial matrices in δ over $S_N, F(\delta, n)$, such that there exists a matrix $F_0(\delta)$ in $\underline{\mathcal{M}_{\mathcal{R}}}$ such that

$$F(\delta, n) = P^n(\delta) F(\delta, 0) P^{-n}(\delta), \tag{2.3}$$

where

$$P(\delta) = \begin{bmatrix} 0 & I_{N-1} \\ \delta & 0 \end{bmatrix}, \qquad P^{-1}(\delta) = \begin{bmatrix} 0 & \delta^{-1} \\ I_{N-1} & 0 \end{bmatrix}$$

with $F(\delta, 0) = F_0(\delta)$.

The reason for defining such a set $\mathcal{M}_{\mathcal{R}}$ is explained by Lemma 3.1 below, which was proved in [12]. The following proposition is obvious.

[2] Here, skew means that the elements of S_N do not commute with the indeterminate q.

[3] As usual, an element of $\Re$ is said to be unimodular iff it is invertible in $\Re$. Note that an unimodular element in $\Re$ is not necessarily an element of S_N; for example, consider the polynomial $f(q) = f_1 q + f_0$ over $\Re := S_2[q]$, with $f_1 \equiv (0, 1), f_0 \equiv (1, 1)$ i.e., $f_1(n) = 0$ (resp. 1) if n is even (resp. odd) and $f_0(n) = 1$ for every n; the polynomial $f(q)$ has an inverse in $\Re$, $f(q)^{-1} = \tilde{f}_1 q + f_0$ where $\tilde{f}_1 \equiv (0, -1)$. One indeed has $f(q) \cdot f(q)^{-1} = f(q)^{-1} \cdot f(q) = 1_S$.

Proposition 2.2. *$\mathscr{M}_{\mathscr{R}}$ is a subring of $\mathscr{M}$.*

3. Algebraic properties of $\mathfrak{R}$

As for a LTI system [18, 15, 2], we will see in Section 4 that the basic algebraic properties of a LPDT system, such as controllability and reachability, are essentially linked to coprimeness. Therefore, the aim of this section is (i) to give a method to study the coprimeness of a pair $(a(q), b(q))$ in $\mathfrak{R}$ via a ring-isomorphism $\mathscr{W}$ from $\mathfrak{R}$ onto $\mathscr{M}_{\mathscr{R}}$ (Proposition 3.1); (ii) to discuss what can be said about coprimeness of a pair $(a(q), b(q))$ in $\mathfrak{R}$ and its associated image $(A(\delta), B(\delta))$ in $\mathscr{M}$ by the ring-isomorphism $\mathscr{W}$, [in particular, it is proven that weak coprimeness over $\mathscr{M}$ and over $\mathscr{M}_{\mathscr{R}}$ are not equivalent (Propositions 3.2 and 3.4)]; (iii) to give a necessary and sufficient condition under which the weak coprimeness of a pair $(a(q), b(q))$ in $\mathfrak{R}$ is equivalent to its strong coprimeness (Theorem 3.1).

Proposition 3.1. *There exists a ring-monomorphism $\mathscr{W}:\mathfrak{R}\to\mathscr{M}$, defined by $\mathscr{W}(f(q))=F(\delta)$, where the matrix $F(\delta)$ is determined by the following relation*:

$$e(q)\cdot f(q)=F(\delta)\cdot e(q), \tag{3.1}$$

[*see an example of construction of $F(\delta)$ from $f(q)$ in Appendix* A.1]. *The image of $\mathfrak{R}$ by $\mathscr{W}$ is $\mathscr{M}_{\mathscr{R}}$ (so that $\mathscr{W}$ can be viewed as a ring-isomorphism from $\mathfrak{R}$ onto $\mathscr{M}_{\mathscr{R}}$).*

Proof. A similar ring-monomorphism $\mathscr{W}$ has been widely used in the transfer matrix approach (see, e.g., [19, 16, 8]); moreover, to see that the image of $\mathfrak{R}$ by $\mathscr{W}$ is $\mathscr{M}_{\mathscr{R}}$, let us recall the following result given in [12] for periodic transfer matrices and also applicable to periodic polynomial matrices.

Lemma 3.1. *For any matrix $F(\delta)$ in $\mathscr{M}$ satisfying* (3.1) *and any integer n,*

$$F(\delta,n+1)=P(\delta)F(\delta,n)P^{-1}(\delta) \tag{3.2}$$

with

$$P(\delta)=\begin{bmatrix}0 & I_{N-1}\\ \delta & 0\end{bmatrix},\qquad P^{-1}(\delta)=\begin{bmatrix}0 & \delta^{-1}\\ I_{N-1} & 0\end{bmatrix}.$$

Remark 3.1. Clearly every matrix in $\mathscr{M}_{\mathscr{R}}$ is entirely defined by its first row, and so are also all coefficients of its inverse image in $\mathfrak{R}$.

Proposition 3.2. *Let $A(\delta)$, $B(\delta)$ be two matrices in $\underline{\mathscr{M}}_{\mathscr{R}}$,*[4] *and assume that the pair $(A(\delta), B(\delta))$ is left* (*resp. right*) *coprime in $\underline{\mathscr{M}}$.*[5] *Then the three following conditions are equivalent*:
(i) *the pair $(A(\delta), B(\delta))$ is strongly left* (*resp. right*) *coprime in $\underline{\mathscr{M}}_{\mathscr{R}}$,*[6]
(ii) *the pair $(A(0), B(0))$ is strongly left* (*resp. right*) *coprime in $\underline{\mathscr{M}}_{\mathscr{R}}$,*
(iii) *for every i, $1\leqslant i\leqslant N$, $(A(0)_{i,i}, B(0)_{i,i})\neq(0,0)$.*

Proof. We consider the left coprimeness only, because for the right one the rationale is completely analogous.

"(i) $\Leftrightarrow$ (ii)". Obviously (i) implies (ii); moreover, the left coprimeness of the pair $(A(\delta), B(\delta))$ in $\underline{\mathscr{M}}$ means that there exist two matrices $X_0(\delta), Y_0(\delta)$ in $\underline{\mathscr{M}}$ such that

$$A(\delta)X_0(\delta)+B(\delta)Y_0(\delta)=I_N \tag{3.3}$$

[4] Recall that such matrices are independent of time.
[5] As $\underline{\mathscr{M}}$ is a principal ideal ring, the notions of strong and weak coprimeness are equivalent and have not to be distinguished.
[6] Left coprimeness in $\underline{\mathscr{M}}_{\mathscr{R}}$ (or in any ring) has a definition similar to that in $\mathfrak{R}$ (see Definition 2.1).

the parametrization of the general solution of Eq. (3.3) is given by [18]

$$\begin{aligned} X(\delta) &= X_0(\delta) + \tilde{B}(\delta)Q(\delta), \\ Y(\delta) &= Y_0(\delta) - \tilde{A}(\delta)Q(\delta), \end{aligned} \tag{3.4}$$

where $Q(\delta)$ is any matrix in $\underline{\mathscr{M}}$, the pair $(\tilde{A}(\delta), \tilde{B}(\delta))$ is right coprime in $\underline{\mathscr{M}}$ and satisfies the relation $A(\delta)\tilde{B}(\delta) = B(\delta)\tilde{A}(\delta)$. By hypothesis the pair $(A(0), B(0))$ is strongly left coprime in $\underline{\mathscr{M}}_{\mathscr{R}}$, i.e., there exist two matrices X_1, Y_1 in $\underline{\mathscr{M}}_{\mathscr{R}}$ such that

$$A(0)X_1 + B(0)Y_1 = I_N. \tag{3.5}$$

Moreover, there exists a matrix Q_1 in $\underline{\mathscr{M}}$ such that

$$X_1 = X_0(0) + \tilde{B}(0)Q_1, \qquad Y_1 = Y_0 - \tilde{A}(0)Q_1.$$

To obtain a solution of Eq. (3.3) in $\underline{\mathscr{M}}_{\mathscr{R}}$, one can choose in Eq. (3.4) any matrix $Q(\delta)$ in $\underline{\mathscr{M}}$ such that $Q(0) = Q_1$.

"(ii) $\Leftrightarrow$ (iii)". Set $\delta = 0$ in Eq. (3.3); it is easy to see that the condition (iii) is necessary for this equation to have a solution in $\underline{\mathscr{M}}_{\mathscr{R}}$; hence, (ii) implies (iii). Conversely, let us prove by induction that (iii) implies (ii). For $N = 1$, $\mathfrak{R} = \mathbb{R}[q]$ and the result is obvious. Now assume that (iii) implies (ii) for some integer N, and let us prove that this property is still true for $N+1$. Let $A(0)$ and $B(0)$ be two upper triangular $(N+1)\times(N+1)$ matrices, hence of the form

$$F(0) = \left[\begin{array}{c|c} f_1 & f_2 \dots f_{N+1} \\ \hline 0 & F_1(0) \end{array}\right] \tag{3.6}$$

where $F(0) = A(0)$ or $B(0)$, and $F_1(0)$ is a $N \times N$ upper triangular matrix. The question is whether there exist two matrices $X(0)$ and $Y(0)$ of the form (3.6), solutions of Eq. (3.5), where N is replaced by $N+1$. This equation is equivalent to the two following ones:

$$A_1X_1 + B_1Y_1 = I_N, \tag{3.7}$$

$$a_1x_i + b_1y_i = -[a_2 \dots a_{N+1}]X_1^{(i-1)} - [b_2 \dots b_{N+1}]Y_1^{(i-1)}, \tag{3.8}$$

where $2 \leqslant i \leqslant N+1$ and $X_1^{(i-1)}$ (resp. $Y_1^{(i-1)}$) is the $(i-1)$th column of $X_1(0)$ (resp. $Y_1(0)$). From the hypothesis of the induction rationale, Eq. (3.7) admits a solution $(X_1(0), Y_1(0))$ in $\underline{\mathscr{M}}_{\mathscr{R}}$. In addition, the N independent equations (3.8) admit always solutions (x_i, y_i), because $(a_1, b_1) \neq (0, 0)$. Hence, the result is proved. □

Remark 3.2. It is clear that condition (iii) of Proposition 3.2 is always satisfied in case of non-reversible systems (in LTI sense); consequently, this condition is not satisfied only for a class of non-reversible systems (by Proposition 3.4 below).

Proposition 3.3. *$\mathscr{M}_{\mathscr{R}}$, and consequently $\underline{\mathscr{M}}_{\mathscr{R}}$ are neither left nor right Bezout rings.*[7]

Proposition 3.4. *Let $(A(\delta), B(\delta))$ be a pair of weakly left (resp. right) coprime matrices in $\mathscr{M}_{\mathscr{R}}$. If this pair is not strongly left (resp. right) coprime in $\mathscr{M}_{\mathscr{R}}$, then the three following conditions hold:*

(i) *There exists at least one integer n, $0 \leqslant n \leqslant N-1$ such that $A(0,n)_{N,N} = B(0,n)_{N,N} = 0$.*

(ii) *There exists at least one integer n, such that pair $(A(\delta,n), B(\delta,n))$ is not left (resp. right) coprime in $\underline{\mathscr{M}}$.*

[7] A counterexample is given in Appendix A.2. Recall that a left (resp. right) Bezout ring is a ring where any finitely generated left (resp. right) ideal is principal.

(iii) *For every integer* n, $C_n(\delta)=gcld(A(\delta,n),B(\delta,n))$ (*resp. gcrd*) *in* $\underline{\mathscr{M}}$ *has a determinant of the form* $\alpha\delta^k$, $\alpha\in\mathbb{R}-\{0\}$, $k>0$.

Proof. This proof is given for the left coprimeness case.

(i) Let $(A(\delta),B(\delta))$ be a pair of weakly but not strongly left coprime matrices in $\mathscr{M}_{\mathscr{R}}$; then, there exists at least one integer i such that the pair $(A(\delta,i),B(\delta,i))$ is weakly left coprime in $\underline{\mathscr{M}}_{\mathscr{R}}$ but not strongly left coprime in $\underline{\mathscr{M}}_{\mathscr{R}}$. So by Proposition 3.2 there exists an integer j, $(1\leqslant j\leqslant N)$ such that $A(0,i)_{j,j}=B(0,i)_{j,j}=0$. Now from Lemma 3.1 we have

$$A(\delta,i+N-j)=P^{N-j}(\delta)A(\delta,i)P^{-(N-j)}(\delta),\qquad B(\delta,i+N-j)=P^{N-j}(\delta)B(\delta,i)P^{-(N-j)}(\delta). \tag{3.9}$$

Moreover, let $C=(C_{i,j})_{N\times N}$ be any upper triangular matrix, and set $D=P(\delta)CP^{-1}(\delta)$. It can be easily verified that $D_{i,i}=C_{i+1,i+1}$ if $1\leqslant i\leqslant N-1$ and $D_{N,N}=C_{1,1}$.

So if $j\neq N$, for $n=i+N-j$ we have

$$A(0,n)_{N,N}=A(0,i)_{j,j}=0\quad\text{and}\quad B(0,n)_{N,N}=B(0,i)_{j,j}=0.$$

(ii) Moreover, as $A(\delta,n)$ and $B(\delta,n)$ belong to $\underline{\mathscr{M}}_{\mathscr{R}}$, we deduce that the last rows of these matrices are multiples of δ. Hence, the matrix $C(\delta)=\mathrm{diag}(1,1,\dots,\delta)$ is a non-unimodular *common left divisor* of $(A(\delta,n),\ B(\delta,n))$ in $\underline{\mathscr{M}}$.

(iii) Let $C_n(\delta)$ be a gcld of $A(\delta,n)$ and $B(\delta,n)$ and write $A(\delta,n)=C_n(\delta)A_n'(\delta)$, $B(\delta,n)=C_n(\delta)B_n'(\delta)$. Using usual elementary column operations, $C_n(\delta)$ can be chosen such that $C_n(0)$ is upper triangular (Hermite form), i.e., belongs to $\underline{\mathscr{M}}_{\mathscr{R}}$.

If the determinant of $C_n(\delta)$ is not a multiple of δ, then $C_n(0)$ is invertible. As a result, $A_n'(0)=C_n(0)^{-1}A(0,n)$ and $B_n'(0)=C_n(0)^{-1}B(0,n)$ are upper triangular, i.e., $A_n'(\delta)$ and $B_n'(\delta)$ belong to $\underline{\mathscr{M}}_{\mathscr{R}}$. Therefore, the pair $(A(\delta),B(\delta))$ cannot be weakly left coprime in $\mathscr{M}_{\mathscr{R}}$.

If the determinant of $C_n(\delta)$ has a non-unit-divisors different from δ, this matrix can be factored out (using, e.g., the Smith form) as $C_n(\delta)=D_n(\delta)E_n(\delta)$, where the determinant of $D_n(\delta)$ is not a unit and is not a multiple of δ and where the determinant of $E_n(\delta)$ is a power of δ. One can write

$$A(\delta,n)=D_n(\delta)A_n''(\delta)\quad\text{and}\quad A(\delta,n)=D_n(\delta)B_n''(\delta).$$

$D_n(\delta)$ is not unimodular and $D_n(0)$ is invertible. By the same rationale as above, in this case again $(A(\delta),B(\delta))$ cannot be weakly left coprime in $\mathscr{M}_{\mathscr{R}}$, and the corollary is proved. □

Theorem 3.1. *Let* $(A(\delta),B(\delta))$ *be a pair of weakly left* (*resp. right*) *coprime matrices in* $\mathscr{M}_{\mathscr{R}}$; *the pair* $(A(\delta),B(\delta))$ *is strongly left* (*resp. right*) *coprime in* $\mathscr{M}_{\mathscr{R}}$ *iff, for all integers* n, *the pair* $(A(0,n),B(0,n))$ *satisfies the condition* (iii) *of Proposition* 3.2.

Proof. This proof is given for the left coprimeness case.

"$\Leftarrow$". If for every n, $0\leqslant n\leqslant N-1$, the pair $(A(0,n),B(0,n))$ satisfies the condition (iii) of Proposition 3.2, then by this Proposition one deduces that the pair $(A(\delta,n),B(\delta,n))$ is strongly left coprime in $\underline{\mathscr{M}}_{\mathscr{R}}$ for every n, and consequently the pair $(A(\delta),B(\delta))$ is strongly left coprime in $\mathscr{M}_{\mathscr{R}}$.

"$\Rightarrow$". Now suppose that $(A(\delta),B(\delta))$ is strongly left coprime in $\mathscr{M}_{\mathscr{R}}$ and that there exists at least one integer n such that $(A(0,n),B(0,n))$ does not satisfy the condition (iii) of Proposition 3.2. From Proposition 3.4 the pair $(A(\delta,n),B(\delta,n))$ is not strongly left coprime in $\underline{\mathscr{M}}_{\mathscr{R}}$; this contradicts the hypothesis that the pair $(A(\delta),B(\delta))$ is strongly left coprime in $\mathscr{M}_{\mathscr{R}}$. □

The following result is immediately clear from Proposition 3.2.

Corollary 3.1. *A pair* $(A(\delta),B(\delta))$, *as in Theorem* 3.1, *is strongly left* (*resp. right*) *coprime in* $\mathscr{M}_{\mathscr{R}}$ *iff it is so in* $\mathscr{M}$.

4. Algebraic properties of LPDT systems

Consider the polynomial description of a monovariable LPDT system $\mathscr{D}$

$$d(q)\xi = n(q)u, \tag{4.1}$$

$$y = t(q)\xi + w(q)u, \tag{4.2}$$

where $d(q)$, $n(q)$, $t(q)$, and $w(q)$ belong to $\mathfrak{R}$, and $d(q)$ is assumed to be non-zerodivisor. [8]

The definition of reachability (resp. observability) used for the propositions below is the complete (h-step) reachability [resp. complete (h-step) observability] defined in [24, 21] in the state space representation, and reformulated in a polynomial representation in terms of coprimeness of polynomial matrices for LTI systems in, e.g., [2, 15].

Theorem 4.1. (i) *$\mathscr{D}$ is reachable iff the pair $(d(q), n(q))$ is strongly left coprime over $\mathfrak{R}$.*
(ii) *$\mathscr{D}$ is observable iff the pair $(d(q), t(q))$ is strongly right coprime over $\mathfrak{R}$.*

Proof. (i) "$\Leftarrow$". If the pair $(d(q), n(q))$ is strongly left coprime, then (by Corollary 3.1) so is the pair $(D(\delta), N(\delta))$, its associated image by the ring-isomorphism $\mathscr{W}$ in $\mathscr{M}_{\mathscr{R}}$. Consequently, the N LTI equivalent systems associated with $\mathscr{D}$ defined by the following two equations are reachable:

$$D(\delta, n)\psi = N(\delta, n)U, \tag{4.3}$$

$$Y = T(\delta, n)\psi + W(\delta, n)U, \tag{4.4}$$

where $0 \leqslant n \leqslant N-1$, $D(\delta) = \mathscr{W}(d(q))$, $N(\delta) = \mathscr{W}(n(q))$, $T(\delta) = \mathscr{W}(t(q))$, $W(\delta) = \mathscr{W}(w(q))$, From Ref. [14] $\mathscr{D}$ is also reachable.

"$\Rightarrow$". Conversely, if $\mathscr{D}$ is reachable then the N-LTI equivalent systems associated with $\mathscr{D}$ and defined by Eqs. (4.3) and (4.4) are reachable [14]. Consequently, for every integer n, $0 \leqslant n \leqslant N-1$, the pair $(D(\delta, n)$, $N(\delta, n))$ is left coprime in $\underline{\mathscr{M}}$. By Corollary 3.1 one deduces that the pair $(d(q), n(q))$ is necessarily strongly left coprime in $\mathfrak{R}$.

By duality the second property of this Theorem can be deduced from the first one. □

Before giving a characterization of controllability and reconstructibility for LPDT systems, recall the following result which concerns discrete LTI systems: consider a LTI system

$$D(\delta)\xi = N(\delta)u, \tag{4.5}$$

$$y = T(\delta)\xi + W(\delta)u, \tag{4.6}$$

where $D(\delta) \in \mathbb{R}^{n\times n}[\delta]$, $N(\delta) \in \mathbb{R}^{n\times m}[\delta]$, $T(\delta) \in \mathbb{R}^{p\times n}[\delta]$ and $W(\delta) \in \mathbb{R}^{p\times m}[\delta]$; $D(\delta)$ is assumed to be full rank, i.e., non-zero divisor.

Lemma 4.1 (Blomberg and Ylinen [2]). (i) *The system* (4.5), (4.6) *is controllable iff the pair $(D(\delta), N(\delta))$ is left coprime over $R[\delta]$.*
(ii) *The system* (4.5), (4.6) *is reconstructible iff the pair $(D(\delta), T(\delta))$ is right coprime over $\mathbb{R}[\delta]$.*

To generalize this result to LPDT systems taken in their periodic PMD, one can try, at first sight, to search for a link with the coprimeness notion over $S_N[q]$. By the example given below, one shows that in this context, the reconstructibility and controllability notions are expressed only by sufficient conditions, and to have necessary and sufficient conditions, we use the fact that the controllability (resp. reconstructibility) criterion of a LPDT system is expressed in terms of necessary and sufficient conditions of controllability

[8] Obviously, a periodic polynomial $a(q)$ is left (or right) non-zerodivisor iff so is $A(q)$, its associated image by $\mathscr{W}$ in $\mathscr{M}_{\mathscr{R}}$.

(resp. reconstructibility) of at least one associated LTI system. We also use the fact that controllability (resp. reconstructibility) of LTI systems is characterized by Lemma 4.1. A new ring – deduced from $\mathfrak{R}$ by a localization [5] – is introduced, in a natural way. The above structural properties are characterizable, over this ring, in terms of weak coprimeness.

For this, let us consider the subset of $\mathcal{M}_{\mathfrak{R}}$ defined by

$$\mathcal{M}_u = \{M \in \mathcal{M}_{\mathfrak{R}} \mid \det(M) = \alpha\delta^n,\ \alpha \in S_N - \{0\},\ n \in \mathcal{N}\}$$

and let $\mathcal{U}$ be the image of $\mathcal{M}_u$ by the ring-isomorphism $\mathcal{W}^{-1}$. Let $\mathfrak{R}_u = \mathcal{U}^{-1} \cdot \mathfrak{R}$ $(= \mathfrak{R} \cdot \mathcal{U}^{-1})$ be the localizing set [5] of $\mathfrak{R}$ in $\mathcal{U}$. By definition, $\mathfrak{R}_u$ is the ring consisting of all elements of the form $a^{-1}b$ or $b'a'^{-1}$, where a and a' belong to $\mathcal{U}$ and b and b' belong to $\mathfrak{R}$.

Controllability and reconstructibility criteria of a LPDT system $\mathcal{D}$ given by its periodic polynomial representation (4.1), (4.2) are given by the following theorem.

Theorem 4.2. (i) *$\mathcal{D}$ is controllable iff the pair $(d(q), n(q))$ is weakly left coprime over $\mathfrak{R}_u$.*
(ii) *$\mathcal{D}$ is reconstructible iff the pair $(d(q), t(q))$ is weakly right coprime over $\mathfrak{R}_u$.*

Proof. (i) "$\Leftarrow$". If the pair $(d(q), n(q))$ is weakly left coprime over $\mathfrak{R}_u$, by construction of $\mathfrak{R}_u$, so is the pair $(D(\delta), N(\delta))$ – its associated image by the ring is isomorphism $\mathcal{W}$ – over $S_N[\delta)$. Hence, from Proposition 3.4, the pair $(D(\delta, n), N(\delta, n))$ is weakly left coprime over $\mathbb{R}[\delta)$ for every integer n. This implies that the N-LTI associated systems of $\mathcal{D}$ are controllable, therefore [11] $\mathcal{D}$ is controllable.

"$\Rightarrow$". Conversely, if $\mathcal{D}$ is controllable then the N-LTI equivalent systems associated with $\mathcal{D}$ and defined by Eqs. (4.3) and (4.4) are controllable [11]. Consequently, for every integer n, $0 \leqslant n \leqslant N-1$, the pair $(D(\delta, n), N(\delta, n))$ is left coprime over $\mathbb{R}[\delta)$. By construction of $\mathfrak{R}_u$ one deduces that the pair $(d(q), n(q))$ is necessarily left coprime over $\mathfrak{R}_u$.

The property (ii) can be deduced from (i) by duality. □

Example. Take the 2-periodic difference equation

$$\{q^3 + d_2 q^2 + d_1 q\} y(k) = \{n_1 q + n_0\} u(k) \tag{4.7}$$

with $d_2 \equiv (2, 1)$, $d_1 \equiv (1, 0)$, $n_1 \equiv (1, 5)$, $n_0 \equiv (1, 0)$ (see footnote 3).

Let us verify at first that Eq. (4.7) defines a (periodic) system i.e., the polynomial $d(q)$ is a non-zero divisor and $d^{-1}(q) \cdot n(q)$ belongs to $S_2[[q^{-1}]]$,[9] where $d(q) = q^3 + d_2 q^2 + d_1 q$ and $n(q) = n_1 q + n_0$.

Let $D(\delta)$ (resp. $N(\delta)$) be the image of $d(q)$ (resp. $n(q)$) by $\mathcal{W}$; one obtains

$$D(\delta) = \begin{bmatrix} d_2\delta & d_1 + \delta \\ d_1^{(1)}\delta + \delta^2 & d_2^{(1)}\delta \end{bmatrix} \quad \text{and} \quad N(\delta) = \begin{bmatrix} n_0 & n_1 \\ n_1^{(1)}\delta & n_0^{(1)} \end{bmatrix}.$$

It is not difficult to see that $D(\delta)$ is a non-zero divisor, thus by applying the ring-isomorphism $\mathcal{W}^{-1}$, one deduces that $d(q)$ is also a non-zero divisor in $\mathfrak{R}$.

By relation (3.1) one can deduce that

$$d^{-1}(q) = \frac{1}{2}[1\ \ q^{-1}] \cdot D^{-1}(\delta) \cdot e(q) = \frac{1}{\delta^2(\delta - 1)}\{q^3 - d_2^{(1)} q^2 + d_1 q\},$$

hence,

$$\begin{aligned} d^{-1}(q) \cdot n(q) &= \frac{1}{\delta^2(\delta - 1)}\{q^3 - d_2^{(1)} q^2 + d_1 q\}\{n_1 q + n_0\} \\ &= \frac{1}{q^4(q^2 - 1)}\{n_1^{(1)} q^4 + (n_0^{(1)} - d_2^{(1)} n_1) q^3 + (d_1 n_1^{(1)} - d_2^{(1)} n_0) q^2 + d_1 n_0^{(1)} q\}, \end{aligned}$$

[9] This condition is imposed to ensure the causality condition for the system defined by Eq. (4.7).

thus $d^{-1}(q) \cdot n(q)$ belongs to $S_2[[q^{-1}]]$. Now we show that system (4.7) is controllable without having the comprimeness of the pair $(d(q), n(q))$ over $S_N[q]$.

The N-LTI systems associated to Eq. (4.7) are given by

$$D(\delta, n)Y = N(\delta, n)U$$

and the matrices $D(\delta)$, $N(\delta)$ can be decomposed in $\mathscr{M}_{\mathfrak{R}}$ as follows:

$$\begin{aligned} D(\delta) &= \begin{bmatrix} d_1 & 1 \\ \delta & d_1^{(1)} \end{bmatrix} \begin{bmatrix} \delta & 1 \\ \delta & \delta \end{bmatrix} = L(\delta)D'(\delta), \\ N(\delta) &= \begin{bmatrix} d_1 & 1 \\ \delta & d_1^{(1)} \end{bmatrix} n_1 = L(\delta)N'(\delta). \end{aligned} \tag{4.8}$$

In addition, $\det(L(\delta,0)) = d_1 d_1^{(1)} - \delta = -\delta = \det(L(\delta,1))$ and the pair $(D'(\delta), N'(\delta))$ is left coprime in $\mathscr{M}$. So the 2-LTI systems associated with Eq. (4.7) (and consequently Eq. (4.7)) are controllable, although $(d(q), n(q))$ is not left coprime over $S_2[q]$; indeed by applying $\mathscr{W}^{-1}$ to Eq. (4.8), one obtains

$$\begin{aligned} d(q) &= \{q + d_1\} q \{q + 1\}, \\ n(q) &= \{q + d_1\} n_1. \qquad \square \end{aligned}$$

5. Concluding remarks

In the present work, the polynomial approach has been developed for the study of algebraic properties of LPDT systems. In the authors' opinion, this approach clarifies the subject. As a matter of fact, it shows that reachability and observability are dependent on time only for a class of non-reversible systems; and consequently, a wide class of LPDT systems are "index invariant". Notice that, in the polynomial description (4.1), (4.2) of $\mathscr{D}$, the only hypothesis is that $d(q)$ is non-zero divisor, hence this study can be applied to LPDT descriptor systems, i.e., those for which the highest order coefficient of $d(q)$ is not invertible in S_N [$d(q)$ is not necessarily an admissible (or monic) polynomial]. According to the usual terminology, such LPDT systems have an order which is dependent on time.

Although we have focused on SISO systems, most of the results presented here have a straightforward extension of MIMO systems.

Our approach has connections with Fliess' module theoretic standpoint [6, 7, 4]. As a matter of fact, from Theorem 4.1, a LPDT system is reachable iff its associated $\mathfrak{R}$-module is projective [22]. In addition, a stronger condition is the freeness of this module, which is equivalent to the existence of a doubly coprime factorization.

These extensions and the use of the module language will be the subject of a forthcoming paper.

Appendix A

For the following examples, the period is $N = 2$, $\mathfrak{R} = S_2[q]$; and for every element a in S_2, set by convention $a \equiv (a(0),\ a(1))$ where $a(i)$ is the value of the sequence a at time i (modulo 2), $i = 0, 1$.

A.1. Image $F(\delta)$ by $\mathscr{W}$ of a polynomial $f(q)$ in $\mathfrak{R}$.

Let $f(q)$ be a polynomial in $\mathfrak{R}$ defined by

$$f(q) = a_0 + a_1 q + \cdots + a_7 q^7, \qquad a_i \in S_2 \text{ and } a_7 \neq 0_S,$$

$$\begin{bmatrix} 1 \\ q \end{bmatrix} \cdot f(q) = \begin{bmatrix} a_0 + a_1 q + \cdots + a_7 q^7 \\ (a_0^{(1)} + a_1^{(1)} q + \cdots + a_7^{(1)} a^7) q \end{bmatrix}$$

$$= \left[\begin{array}{c|c} a_0 + a_2\delta + a_4\delta^2 + a_6\delta^3 & a_1 + a_3\delta + a_5\delta^2 + a_7\delta^3 \\ \hline a_1^{(1)}\delta + a_3^{(1)}\delta^2 + a_5^{(1)}\delta^3 + a_7^{(1)}\delta^4 & a_0^{(1)} + a_2^{(1)}\delta + a_4^{(1)}\delta^2 + a_6^{(1)}\delta^3 \end{array}\right] \begin{bmatrix} 1 \\ q \end{bmatrix}$$

and

$$F(\delta) = \left[\begin{array}{c|c} a_0 + a_2\delta + a_4\delta^2 + a_6\delta^3 & a_1 + a_3\delta + a_5\delta^2 + a_7\delta^3 \\ \hline a_1^{(1)}\delta + a_3^{(1)}\delta^2 + a_5^{(1)}\delta^3 + a_7^{(1)}\delta^4 & a_0^{(1)} + a_2^{(1)}\delta + a_4^{(1)}\delta^2 + a_6^{(1)}\delta^3 \end{array}\right].$$

A.2. A counterexample to show that $\mathfrak{R}$ is not a Bezout ring, i.e., a pair $(f_1(q),\ f_2(q))$ in $\mathfrak{R}$ is defined below, which is weakly left coprime but not strongly left coprime.

Take $f_1(q) = \alpha_1 q^2 + \beta_1$ and $f_2(q) = \alpha_2 q^2 + \beta_2 q$ with $\alpha_1 \equiv (1,0)$, $\beta_1 \equiv (0,1)$, $\alpha_2 \equiv (0,3)$, $\beta_2 \equiv (2,1)$, and take the images $F_1(\delta)$ (resp. $F_2(\delta)$) of $f_1(q)$ (resp. $f_2(q)$) by the ring-isomorphism $\mathscr{W}$. One obtains

$$F_1(\delta) = \begin{bmatrix} \alpha_1\delta + \beta_1 & 0 \\ 0 & \alpha_1^{(1)}\delta + \beta_1^{(1)} \end{bmatrix}, \qquad F_2(\delta) = \begin{bmatrix} \alpha_2\delta & \beta_2 \\ \beta_2^{(1)}\delta & \alpha_2^{(1)}\delta \end{bmatrix}$$

and

$$F_1(\delta,0) = \begin{bmatrix} \delta & 0 \\ 0 & 1 \end{bmatrix}, \qquad F_2(\delta,0) = \begin{bmatrix} 0 & 2 \\ \delta & 3\delta \end{bmatrix}.$$

The pair $(F_1(\delta,0), F_2(\delta,0))$ is left coprime in $\mathscr{M}$, indeed, with the usual column transformations, one can calculate matrices $X(\delta), Y(\delta)$ in $\mathscr{M}$

$$X(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad Y(\delta) = \begin{bmatrix} -3/2 & 0 \\ 1/2 & 0 \end{bmatrix}$$

such that

$$F_1(\delta,0)X(\delta) + F_2(\delta,0)Y(\delta) = I_2. \tag{A.1}$$

Moreover,

$$F_1(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad F_2(0,0) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

and, from Proposition 3.2, Eq. (A.1) has a solution in $\mathscr{M}_{\mathscr{R}}$ iff there exist two matrices of the form $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ and $\begin{bmatrix} x' & y' \\ 0 & z' \end{bmatrix}$ over R such that

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x' & y' \\ 0 & z' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Obviously this is impossible, so Eq. (A.1) has a solution in $\mathscr{M}$ but no solution in $\mathscr{M}_{\mathscr{R}}$.

Now suppose that the pair $(f_1(q), f_2(q))$ is not weakly left coprime, i.e., there exists a non unimodular polynomial $d(q)$ in $\mathfrak{R}$, such that

$$f_1(q) = d(q) \cdot f_1'(q) \quad \text{and} \quad f_2(q) = d(q) \cdot f_2'(q),$$

consequently,

$$F_1(\delta) = D(\delta) \cdot F_1'(\delta) \quad \text{and} \quad F_2(\delta) = D(\delta) \cdot F_2'(\delta),$$

in particular,

$$F_1(\delta,1) = D(\delta,1) \cdot F_1'(\delta,1) \quad \text{and} \quad F_2(\delta,1) = D(\delta,1) \cdot F_2'(\delta,1). \tag{A.2}$$

From Lemma 3.1 and the relation (A.2) one can deduce

$$F_1(\delta,0)=P^{-1}(\delta)D(\delta,1)P(\delta)\cdot P^{-1}(\delta)F_1'(\delta,1)P(\delta)$$

and

$$F_2(\delta,0)=P^{-1}(\delta)D(\delta,1)P(\delta)\cdot P^{-1}(\delta)F_2'(\delta,1)P(\delta).$$

The pair $(F_1(\delta,0),\ F_2(\delta,0))$ is left coprime in $\mathscr{M}$, so the matrix

$$P^{-1}(\delta)D(\delta,1)P(\delta)$$

is necessarily unimodular, in particular $\det(P^{-1}(\delta)D(\delta,1)P(\delta))$ is a scalar and so is $\det(D(\delta,1))$; this implies that $D(\delta,1)$ is an unimodular in $\underline{\mathscr{M}}_{\mathscr{R}}$. Thus, the pair $(f_1(q),f_2(q))$ is weakly but not strongly left coprime. This completes the proof that $\mathfrak{R}$ is not a Bezout ring.

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