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Poles and Zeros at Infinity of Linear Time-Varying Systems

Henri Bourlès and Bogdan Marinescu

***Abstract*—The notions of poles and zeros at infinity and their relations are extended to the case of linear continuous *time-varying systems*. This study is based on the notion of a "newborn system" which is, in a mathematical point of view, a graded module extension over the noncommutative ring of differential operators. It is proved to be a relevant generalization to the time-varying case of the equivalence class, for the so-called "restricted equivalence" of Rosenbrock's polynomial matrix descriptions. The authors' approach is intrinsic and unifies the definitions previously given in the literature in the time-invariant case.**

***Index Terms*— Module, noncommutative rings, structure at infinity, time-varying systems.**

I. Introduction

Poles and zeros at infinity of linear time-invariant systems have been extensively studied since the end of the 1970's (see, e.g., [1], [10], [13], [27]–[30], [24], [10], [7], [21], [23], and [22] for a comprehensive treatment). The *system poles at infinity* consist of the *transmission poles at infinity* and of the *hidden modes at infinity* [27]. The transmission poles at infinity are related to the number of differentiations between the input and the output. The hidden modes at infinity are related to the impulsive motions which can arise inside a system formed at an initial time (due to a failure or a switch) with arbitrary initial conditions and which cannot be eliminated with a nondistributional input or which cannot be observed [27]; those impulsive motions are due to the "compliance constraints" [32] when they are violated. Such a system is called a *newborn system* in the sequel (where this notion is mathematically defined). The *system zeros at infinity* consist of the *transmission zeros at infinity* and of the hidden modes at infinity [13]. The transmission zeros are related to the number of integrators between the input and the output, i.e., to the *relative degree* of the transfer matrix. The problem of impulsive motions or of properness of linear time-varying systems has been recently tackled in [16], [11], and [31], but the complete theory of the structure at infinity of such systems (or more specifically, of linear time-varying *newborn systems*) is developed here for the first time (preliminary results have been presented in [3]–[6]).

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Our approach is based on the theory of the noncommutative principal ideal domains and of modules over such rings [8]. It is strongly connected with the module-based framework developed by Fliess (see [14], [15], and related references). Fliess defined a (time-varying) *linear system* as being a module. This can be explained as follows [2]. Consider, for example, two Rosenbrock's polynomial matrix descriptions (PMD's). They are "strictly equivalent" [20] (hence, as far as the behavior at finite frequencies is concerned, they can be considered as describing the same system) if, and only if (iff) the associated modules are isomorphic; as a result, modules are well suited for an "intrinsic" description of linear systems at finite frequencies. This approach has been used in [2] for studying finite poles and zeros of linear time-invariant systems in an "intrinsic" manner, and the present paper completes [2].

As is well known, structure at infinity is lost when using strict equivalence [27]. For this reason, "restricted equivalence" of PMD's was introduced in [1] (in a wider sense than the one originally proposed by Rosenbrock); it slightly generalizes the notion of "strong equivalence" introduced in [27]. For the same reason, in order to characterize structure at infinity, Fliess' approach is completed here and the notion of "linear newborn system" is introduced. It is a "module extension" by graded free modules, or for short, a "graded module extension." As is shown below, two PMD's are restrictly equivalent iff they are the representations, in different bases, of the same newborn system. As in the usual time-invariant case, structural indexes, order, and degree (i.e., the various kinds of multiplicities) of a pole or a zero at infinity are defined and characterized here (generalizing the terminology used in, e.g., [18]); but such a pole or a zero is defined as being a module, as in, e.g., [10], [21], and [23].

II. Mathematical Tools

A. Some Noncommutative Rings [8]

Let $\boldsymbol{K} \supseteq \mathbb{R}$ be a ground differential field, i.e., a commutative field equipped with a derivation denoted by "$\dot{}$" $\boldsymbol{R} := \boldsymbol{K}[s]$ denotes the ring of polynomials with coefficients in $\boldsymbol{K}$ and indeterminate s; the latter has the meaning of the usual derivation, and $\boldsymbol{R}$ is equipped with the following "commutation rule": for every a in $\boldsymbol{K}$

$$sa = as + \dot{a}. \tag{1}$$

Right-multiplying (1) by a time function, it appears to be the usual Leibniz rule. In other words, $\boldsymbol{R}$ is the ring of differential operators with coefficients in $\boldsymbol{K}$, the field of (possibly) time-varying coefficients.

The ring $\boldsymbol{R}$ is a (left and right) principal ideal domain. In addition, $\boldsymbol{R}$ is a left and right Ore domain, hence its field of left fractions and its field of right fractions exist and coincide; this quotient field is denoted by $\boldsymbol{F}$.

Set $\sigma = 1/s$ (so that σ can be viewed as the "integration operator"). Consider the ring $\boldsymbol{S} := \boldsymbol{K}[[\sigma]]$ of formal power series in σ, i.e., consisting of elements $\boldsymbol{a}$ of the form

$$\boldsymbol{a} = a_0 + a_1\sigma + a_2\sigma^2 + \cdots. \tag{2}$$

This ring is equipped with the commutation rule deduced from (1)

$$\sigma a = a\sigma - \sigma \dot{a} \sigma \tag{3}$$

which is the rule of "integration by parts." The ring $\boldsymbol{S}$ has the following properties.

1) An element $\boldsymbol{a}$ of $\boldsymbol{S}$, of the form (2), is a unit (i.e., is invertible in $\boldsymbol{S}$), iff $a_0 \neq 0$.
2) Set $\omega(\boldsymbol{a}) = \min\{j: a_j \neq 0\}$; the integer $\omega(\boldsymbol{a})$ is called the *order* of $\boldsymbol{a}$, and $\boldsymbol{a}$ can be put into the form $\boldsymbol{a} = \upsilon\sigma^{\omega(\boldsymbol{a})} = \sigma^{\omega(\boldsymbol{a})}\upsilon'$, where υ and υ' are units.
3) Therefore, $\boldsymbol{S}$ is a (left and right) principal ideal domain, commutative iff $\boldsymbol{K}$ is a field of constants (i.e., of elements whose derivative is zero), and every nonzero ideal of $\boldsymbol{S}$ is of the form $\sigma^k\boldsymbol{S} = \boldsymbol{S}\sigma^k := (\sigma^k)$. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two nonzero elements of $\boldsymbol{S}$; $\boldsymbol{b}$ divides $\boldsymbol{a}$ (right and left) iff $\omega(\boldsymbol{b}) \leq \omega(\boldsymbol{a})$.
4) The ring $\boldsymbol{S}$ has a quotient field, which is the field $\boldsymbol{L} := \boldsymbol{K}((\sigma))$ of Laurent series in σ, and the quotient field $\boldsymbol{F}$ of $\boldsymbol{R}$ can be embedded in $\boldsymbol{L}$ (in other words, every element of $\boldsymbol{F}$ can be considered as an element of $\boldsymbol{L}$, which is of the form $\Sigma_{i\geq\nu}\, a_i\sigma^i$, $\nu \in \mathbb{Z}$, $a_\nu \neq 0$).

B. Matrices over $\boldsymbol{S}$ and over $\boldsymbol{L}$

The set of unimodular (i.e., square and invertible) matrices over $\boldsymbol{S}$ and of dimension $n \times n$ is denoted by $\boldsymbol{U}_n$. The proof of the following result is straightforward and is detailed in [5].

Proposition 1: Let

$$P = \sum_{i=0}^{\infty} \Gamma_i \sigma^i$$

be an element of $\boldsymbol{S}^{n\times n}$, where $\Gamma_i \in \boldsymbol{K}^{n\times n}$, $i \in \mathbb{N}$. Then, $P \in \boldsymbol{U}_n$ iff Γ_0 is invertible. In this case, for every $k \in \mathbb{N}$, there exist matrices P_k and P_k' in $\boldsymbol{U}_n$ such that $\sigma^k P = P_k\sigma^k$ and $P\sigma^k = \sigma^k P_k'$.

Even in the noncommutative case, the "Smith form" of a matrix $A \in \boldsymbol{S}^{n\times m}$ exists and is

$$\begin{bmatrix} \operatorname{diag}\{\sigma^{\mu_i}\} & 0 \\ 0 & 0 \end{bmatrix}$$

$1 \leq i \leq r$, where $0 \leq \mu_1 \leq \cdots \leq \mu_r$; it is easy to show that this form can be obtained using the three classic elementary column and row operations. As in the usual commutative case, the σ^{μ_i}, $1 \leq i \leq r$, are called the *invariant factors* of A [8].

Definition 1: The μ_i, $1 \leq i \leq r$, are called the *structural indexes* of A, μ_r is called its *order*, and $\mu_1 + \cdots + \mu_r$ is called its *degree*.

This definition is consistent with the terminology of [18]. Let us now generalize the classic Smith–MacMillan form to matrices with entries in $\boldsymbol{L}$. Let $H = H(\sigma)$ be such a matrix.

Proposition 2: There exist unimodular matrices P_k' and U over $\boldsymbol{S}$ such that

$$P_k'^{-1} H U = \begin{bmatrix} \operatorname{diag}\{\sigma^{\nu_i} & 0 \\ 0 & 0 \end{bmatrix}$$

$1 \leq i \leq r$, where $\nu_1 \leq \cdots \leq \nu_r$. The integers ν_i, $1 \leq i \leq r$, are uniquely defined from H. (The above matrix is called the Smith–MacMillan form of H over $\boldsymbol{L}$.)

Proof: Let σ^k be the least common denominator of all entries of H. Then, H can be written $H(\sigma) = \sigma^{-k}A(\sigma)$, where $A = A(\sigma)$ is a matrix with entries in $\boldsymbol{S}$. Let P and U be unimodular matrices over $\boldsymbol{S}$ such that $P^{-1}AU$ is the Smith form of A. By Proposition 1, $P_k'^{-1}HU = P_k'^{-1}\sigma^{-k}AU = (\sigma^k P_k')^{-1}AU = (P\sigma^k)^{-1}AU = \sigma^{-k}P^{-1}AU$. Clearly, this matrix is the Smith–MacMillan form of H over $\boldsymbol{L}$ and $\nu_i = \mu_i - k$, $1 \leq i \leq r$. □

Consider now a matrix $G = G(s)$ with entries in $\boldsymbol{F}$ (i.e., the transfer matrix of a linear time-varying system [15]). As stated above, $\boldsymbol{F}$ can be embedded in $\boldsymbol{L}$, so that $G(\sigma^{-1}) = H(\sigma)$ can be considered as a matrix with entries in $\boldsymbol{L}$. The Smith-MacMillan form of H over $\boldsymbol{L}$ completely describes the structure at infinity of G. The following notions are usual in the case $\boldsymbol{K} = \mathbb{R}$ [18], [26] and are now generalized to the case of any differential ground field.

Definition 2: The integers ν_i, $1 \leq i \leq r$, are the *structural indexes* of $G(s)$ at infinity. If $\nu_1 < 0$, $-\nu_1$ is the *order* of the *pole* of $G(s)$ at infinity, denoting by Σ_p the sum of all negative ν_i, $-\Sigma_p$ is the *degree* of the *pole* of $G(s)$ at infinity. Similarly, if $\nu_r > 0$, this integer is the *order* of the *zero* of $G(s)$ at infinity, and denoting by Σ_z the sum of all positive ν_i, Σ_z is the *degree* of the zero of $G(s)$ at infinity. If $\nu_1 > 0$, then $G(s)$ is said to have a *blocking zero at infinity* with order ν_1.

Obviously, $G(s)$ can be expanded as $G(s) = \Sigma_{i=\nu_1}^{\infty}\, \Theta_i\sigma^i$ where $\Theta_{\nu_1} \neq 0$. Hence, the transfer matrix $G(s)$ is proper (respectively, *strictly proper*) iff $\nu_1 \geq 0$ (respectively, $\nu_1 \geq 1$), and the *index* of $G(s)$ is max (0, 1 $-\nu_1$) [19], [15], [16], [11]; $G(s)$ is biproper iff it is invertible, proper, and with a proper inverse.

C. Modules

Some basic results about finitely generated modules over principal ideal domains are recalled here. For a more detailed intuitive introduction of these notions (in the commutative case), see [2] where the connection with Rosenbrock's PMD's is also widely developed.

1) Let $\boldsymbol{D}$ be a (not necessarily commutative) principal (left and right) ideal domain (e.g., $\boldsymbol{D} = \boldsymbol{R}$ or $\boldsymbol{S}$) and $\boldsymbol{w} = \{w_1, \cdots, w_q\}$ be a finite subset of a left $\boldsymbol{D}$-module M. The column matrix $[w_1, \cdots, w_q]^T$ and the submodule spanned by $\boldsymbol{w}$ are, respectively, written w and $[w]_{\boldsymbol{D}}$ (the module generated by the empty subset of M is the trivial submodule consisting of zero alone, and is denoted by zero). *All modules considered here are finitely generated modules over left and right principal ideal domains having the left and right Ore property.* The properties of such modules recalled below are well known [8].
2) For every $\boldsymbol{D}$-module M, there exists a short exact sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{\ f\ } \mathcal{F} \xrightarrow{\ \phi\ } M \longrightarrow 0 \tag{4}$$

where $\mathcal{E}$ and $\mathcal{F}$ are free $\boldsymbol{D}$-modules; (4) is called a *presentation* of M; the triple $M^* = (f, \mathcal{E}, \boldsymbol{F})$ is called an *extension* of the $\boldsymbol{D}$-module M by the $\boldsymbol{D}$-modules $\mathcal{E}$ and $\mathcal{F}$. All extensions considered here are extensions by free modules. Let $\{\epsilon_1, \cdots, \epsilon_q\}$ and $\underline{\boldsymbol{w}} = \{\underline{w}_1, \cdots, \underline{w}_k\}$ be bases of $\mathcal{E}$ and $\mathcal{F}$, respectively. In these bases, f is represented by a matrix S; S is called a *matrix of definition* of M (or of its extension M^*). Set $e_i = f(\epsilon_i)$, $1 \leq i \leq q$, so that $e = S^T\underline{w}$, and let $w_i = \phi(\underline{w}_i)$, $1 \leq i \leq k$; then, $M = [w]_{\boldsymbol{D}} \cong [\underline{w}]_{\boldsymbol{D}}/[e]_{\boldsymbol{D}}$ and one has

$$S^T w = 0. \tag{5}$$

Equation (5) is called the equation of the module M [2] (or of its extension M^*) in the chosen bases.
3) Let $\boldsymbol{Q}$ be the quotient field of $\boldsymbol{D}$. The extension of the ring of scalars from $\boldsymbol{D}$ to $\boldsymbol{Q}$ is the functor $\boldsymbol{Q} \otimes_{\boldsymbol{D}} -$. Let M be a $\boldsymbol{D}$-module, and set $\check{M} = \boldsymbol{Q} \otimes_{\boldsymbol{D}} M$ and, for any element m of M, $\check{m} = 1_{\boldsymbol{Q}} \otimes m$ ($\check{M}$ is a $\boldsymbol{Q}$-vector space). Let $m_1, \cdots, m_q$ be elements of M; they are $\boldsymbol{D}$ linearly independent iff $\check{m}_1, \cdots, \check{m}_q$ are $\boldsymbol{Q}$ linearly independent. In particular, an element m is torsion iff $\check{m} = 0$. A $\boldsymbol{D}$-module M can be written as a direct sum $M = \mathcal{T}(M) \oplus \Phi$, where $\mathcal{T}(M)$ is the torsion submodule of M and where $\Phi \cong M/\mathcal{T}(M)$ is a free submodule (unique up to isomorphism). The *rank* of M, written $rk(M)$, is the rank of Φ, i.e., the cardinality of any basis of Φ [so that $\Phi \cong \boldsymbol{D}^\rho$,

where $\rho := rk(\Phi)$]; this rank is equal to the dimension of the vector space $\check{M}$. Let M_1 and M_2 be two modules such that $M_1 \subseteq M_2$; one has $\check{M}_1 = \check{M}_2$ iff M_2/M_1 is torsion.

Consider now the case $\boldsymbol{D} = \boldsymbol{R}$ (hence $\boldsymbol{Q} = \boldsymbol{F}$). Then, according to Fliess [18], $\boldsymbol{F}\otimes_{\boldsymbol{R}}-$ is called the *Laplace functor*; it is a generalization to the time-varying case of the usual Laplace transform (with zero initial conditions). A *linear system* is a $\boldsymbol{R}$-module Λ [14], and $\mathcal{F} \otimes_{\boldsymbol{R}} \Lambda = \hat{\Lambda}$ is called the *transfer vector space* of Λ [15]. Note that, as $\mathcal{F} \subset \boldsymbol{L}$, $\boldsymbol{L} \otimes_{\boldsymbol{R}} \Lambda = \boldsymbol{L} \otimes_{\boldsymbol{F}} \hat{\Lambda}$.

Consider the case $\boldsymbol{D} = \boldsymbol{S}$. The invariant factors of any matrix of definition S of an $\boldsymbol{S}$-module M^+ are only dependent on M^+. Therefore, we are led to the following definition.

Definition 3: The structural indexes, the order and the degree of M^+, are those of any matrix of definition of this module.

4) Graded modules: Modules over $\boldsymbol{R}$ can be considered as being graded. To explain this, let us take the example of $\mathcal{E}$. Let $\mathcal{E}_0$ be the $\boldsymbol{K}$-vector space spanned by $\{\epsilon_1, \cdots, \epsilon_q\}$, and set $\mathcal{E}_i = s^i\mathcal{E}_0,\ i \geq 1$. Considering the $\mathcal{E}_i,\ i \geq 0$, and $\mathcal{E}$ as Abelian groups, one can write $\mathcal{E} = \oplus_{i\geq 0}\ \mathcal{E}_i$. The module $\mathcal{E}$, equipped with this structure, is said to be a *graded module*. A change of basis in the graded free module $\mathcal{E}$ is a "graded automorphism," represented by an invertible matrix U over $\boldsymbol{K}$ (i.e., such a transformation matrix U is *not* a polynomial matrix). For convenience, a module extension $\Lambda^* = (f, \mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ are graded as above, is called a *graded module extension.*

III. Newborn Dynamics and Restricted Equivalence

A. Newborn System

Consider the PMD with time-varying coefficients [17]

$$\begin{aligned} D(s)\xi &= N(s)u \\ y &= Q(s)\xi + W(s)u \end{aligned} \tag{6}$$

where $D(s) \in \boldsymbol{R}^{n\times n}$, $N(s) \in \boldsymbol{R}^{n\times m}$, $Q(s) \in \boldsymbol{R}^{p\times n}$, $W(s) \in \boldsymbol{R}^{p\times m}$; the column matrices u, ξ, and y are the input, the partial state, and the output, and they are of lengths m, n, and p, respectively. It is assumed that $D(s)$ is full rank over $\boldsymbol{F} := \boldsymbol{K}(s)$. Equations (6) can be written in a form similar to (5)

$$\underbrace{\begin{bmatrix} D(s) & -N(s) & 0 \\ Q(s) & W(s) & -I_p \end{bmatrix}}_{S^T(s)} \underbrace{\begin{bmatrix} \xi \\ u \\ y \end{bmatrix}}_{\boldsymbol{w}} = 0. \tag{7}$$

As was said above, $S(s)$ is a matrix of definition of a module extension $\Lambda^* = (f, \mathcal{E}, \mathcal{F})$ from which the $\boldsymbol{R}$-module $\Lambda \cong \operatorname{coker} f$ is defined up to isomorphism. If a change of basis is made in $\mathcal{E}$ and $\mathcal{F}$, the matrix of definition $S(s)$ is changed, whereas the module extension Λ^* is left unchanged; the matrix $S^T(s)$ is now replaced by

$$S'^T = US^TV \tag{8}$$

where the matrices $U = U(s)$ and $V = V(s)$ are unimodular over $\boldsymbol{R}$. Clearly, $S(s)$ and $S'(s)$ are matrices of definition of the same linear system. However, the structure at infinity of $S(s)$ is the same as that of $S'(s)$ iff U and V are biproper [26], i.e., are invertible matrices over $\boldsymbol{K}$. Such matrices correspond to particular changes of bases which preserve the orders of differentiations of the variables in (5). From Section II-C4, this is related to the notion of *grading*. We are led to the following definition.

Definition 4: A *newborn linear system* Λ^* is a graded module extension over $\boldsymbol{R}$.

B. Construction of the $\boldsymbol{S}$-Module Λ^+

There exists a left-coprime factorization $(A(\sigma), B(\sigma))$ of $S^T(1/\sigma)$ over $\boldsymbol{S}$ [5]

$$S^T(1/\sigma) = A^{-1}(\sigma)B(\sigma) \tag{9}$$

The matrix $B(\sigma)$ of (9) is a matrix of definition of a $\boldsymbol{S}$-module Λ^+ and the following result is obvious (see [6] for details).

Proposition 3: The $\boldsymbol{S}$-module Λ^+ is uniquely defined from the newborn system Λ^*, up to isomorphism.

Therefore, the S-module Λ^+ is determined by calculating $B(\sigma)$, which is one of its definition matrices. An equation of Λ^+ is

$$B^T(\sigma)w^+ = 0 \tag{10}$$

(see [6] for an abstract construction of Λ^+).

C. Newborn Dynamics

1) According to Fliess [14], a *linear dynamics* $\mathcal{D}$ is a $\boldsymbol{R}$-module Λ (i.e., a linear system) where an input $\boldsymbol{u}$ (with m elements) and (possibly) an output $\boldsymbol{y}$ (with p elements) have been chosen such that $\mathcal{D}/[u]_{\boldsymbol{R}}$ is torsion. The input $\boldsymbol{u}$ is said to be *independent* iff the module $[u]_{\boldsymbol{R}}$ is free of rank m. In this case, there exists a unique matrix $G(s) \in \boldsymbol{F}^{p\times m}$ such that $\hat{y} = G(s)\hat{u}$ and $G(s)$ is the *transfer matrix* of $\mathcal{D}$ [15]. The definition of a *newborn linear dynamics* can now be given.

Definition 5: A newborn linear dynamics $\mathcal{D}^*$ is a newborn system $(f, \mathcal{E}, \mathcal{F})$ such that an "input" $\underline{\boldsymbol{u}} = \{\underline{u}_1, \cdots, \underline{u}_m\}$ and an "output" $\underline{\boldsymbol{y}} = \{\underline{y}_1, \cdots, \underline{y}_p\}$ have been chosen in $\mathcal{F}$ such that $\underline{\boldsymbol{u}} \cup \underline{\boldsymbol{y}}$ is free and $[\underline{\boldsymbol{u}}, \underline{\boldsymbol{y}}]_{\boldsymbol{R}}$ is a direct summand of $\mathcal{F}$ (i.e., $\mathcal{F}$ is of the form $\Xi \oplus [\underline{\boldsymbol{u}}, \underline{\boldsymbol{y}}]_{\boldsymbol{R}}$), $\mathcal{D}/[u]_{\boldsymbol{R}}$ is torsion, and $[u]_{\boldsymbol{R}}$ is free of rank m.

The variable w^+ of (10) can then be written [according to (7)]

$$w^+ = [\xi^{+T} \quad u^{+T} \quad y^{+T}]^T. \tag{11}$$

2) Examples of newborn linear dynamics and of the associated $\boldsymbol{S}$-module $\mathcal{D}^+$.

In all examples below, $\boldsymbol{K} = \mathbb{R}(t)$ where, roughly speaking, $t \geq t_0$ denotes the time.

Example 1: In this example, we consider the PMD with time-varying coefficients

$$\begin{aligned} \xi_1 &= 0 \\ ts^3\xi_1 + s^2\xi_2 &= (t-1)su \\ y &= ts\xi_1 + t^2su. \end{aligned}$$

Note that the first equation cannot be replaced in the two following ones, due to the nonzero initial conditions $\xi_1(t_0^-)$, $\dot{\xi}_1(t_0^-)$,, etc. Put this PMD into the form (7). The graded free modules $\mathcal{E}$ and $\mathcal{F}$ are of ranks 3 and 4, respectively, the basis chosen in $\mathcal{E}$ (respectively, $\mathcal{F}$) is written $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ (respectively, $\{\underline{\xi}_1, \underline{\xi}_2, \underline{u}, \underline{y}\}$) and the morphism f is defined by $f(\epsilon_1) = \underline{\xi}_1$, $f(\epsilon_2) = ts^3\underline{\xi}_1 + s^2\underline{\xi}_2 + (1-t)s\underline{u}$, $f(\epsilon_3) = ts\underline{\xi}_1 + t^2s\underline{u} - \underline{y}$. Obviously, $\mathcal{D}/[u]_{\boldsymbol{R}}$ is torsion, hence $\mathcal{D}^* = (f, \mathcal{E}, \mathcal{F})$, with input $\underline{u}$ and $\underline{y}$ output $\underline{y}$, is a newborn linear dynamics.

The left-coprime factorization (9) is obtained as usual [18], provided that the rule (3) is systematically used; one obtains

$$A(\sigma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma^3 & 0 \\ 0 & 0 & \sigma \end{bmatrix}$$

$$B(\sigma) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t-3\sigma & \sigma & 3\sigma^3 + (1-t)\sigma^2 & 0 \\ t-\sigma & 0 & t^2 - 2\sigma t & -\sigma \end{bmatrix}.$$

Therefore, the $\boldsymbol{S}$-module $\mathcal{D}^+$ is defined by

$$\begin{aligned}\xi_1^+ &= 0\\ (t-3\sigma)\xi_1^+ + \sigma\xi_2^+ + [3\sigma^3 + (1-t)\sigma^2]u^+ &= 0\\ (t-\sigma)\xi_1^+ + [t^2 - 2\sigma t]u^+ - \sigma y^+ &= 0.\end{aligned}$$

The first equation can now be replaced in the two following ones.

Example 2: Consider the "kth-order time-varying derivator" $y = ts^k u$. The same rationale shows that the $\boldsymbol{S}$-module $\mathcal{D}^+$ is defined by $\sigma^k y^+ = (t - k\sigma)u^+$.

Example 3: Consider the "kth-order time-varying integrator" $ts^k y = u$. $\mathcal{D}^+$ is defined by $(t - k\sigma)y^+ = \sigma^k u^+$.

D. Restricted Equivalence

Definition 6: Consider two PMD's of the form (6), with the same inputs and outputs. They are said to be strictly equivalent iff they are the representations, in different bases, of the same newborn dynamics.

The following result proves that the above definition of restricted equivalence is consistent with that given in [1] and [27] in the time-invariant case.

Theorem 1: Let $[D(s), N(s), Q(s), W(s)]$ and $[D'(s), N'(s), Q'(s), W'(s)]$ be two PMD's as in (6). They are restrictly equivalent iff there exist matrices $T, X, R,$ and Y over $\boldsymbol{K}$, of sizes $n\times n$, $p\times n$, $n\times n$, and $n\times m$, respectively, such that T and R are invertible and such that

$$\begin{bmatrix} D'(s) & -N'(s)\\ Q'(s) & W'(s)\end{bmatrix} = \begin{bmatrix} T & 0\\ X & I_p\end{bmatrix}\begin{bmatrix} D(s) & -N(s)\\ Q(s) & W(s)\end{bmatrix}\begin{bmatrix} R & Y\\ 0 & I_m\end{bmatrix}. \tag{12}$$

Proof: Define $S^T(s)$ and $S'^T(s)$ according to (7). There exist two invertible matrices U and V over $\boldsymbol{K}$ such that (8) holds. The transformation (8) is compatible with the structure of $S^T(s)$ and $S'^T(s)$ and the basis $\underline{\boldsymbol{u}} \cup \underline{\boldsymbol{y}}$ of $[\underline{\boldsymbol{u}}, \underline{\boldsymbol{y}}]_{\boldsymbol{R}}$ is left unchanged (see Definition 5) iff the matrices U and V are of the form

$$U = \begin{bmatrix} T & 0\\ X & I_p\end{bmatrix}, \qquad V = \begin{bmatrix} R & Y & 0\\ 0 & I_m & 0\\ 0 & 0 & I_p\end{bmatrix}.$$

Clearly, (8) is then equivalent to (12). □

IV. Poles and Zeros at Infinity

A. Definitions and Relations

In this paper, poles and zeros at infinity are $\boldsymbol{S}$-modules.[1] Their structural indexes, orders, and degrees are defined according to Definition 3. These poles and zeros are defined by analogy with the modules of finite poles and zeros defined in [2].

Remark: In what follows, $\mathcal{T}^+$ denotes $\mathcal{T}(\mathcal{D}^+)$, Φ^+ is a free submodule of $\mathcal{D}^+$ isomorphic to $\mathcal{D}^+/\mathcal{T}^+$ (see Section II-C3), and, by a slight abuse of notation, the projection $\mathcal{D}^+ \to \mathcal{D}^+/\mathcal{T}^+$ is identified with the projection $\mathcal{D}^+ \to \Phi^+$. In addition, $[u^+]_{\boldsymbol{S}}$ is free (because so is $[u]_{\boldsymbol{R}}$), hence it is identified with a submodule of Φ^+ (still denoted by $[u^+]_{\boldsymbol{S}}$); see [2].

Definition 7: The various poles and zeros at infinity of a newborn linear dynamics $\mathcal{D}^*$ are the following $\boldsymbol{S}$-modules:

1) input-decoupling zero at infinity (i.d.z. at ∞): $\mathcal{D}^+$;
2) output-decoupling zero at infinity (o.d.z. at ∞): $\mathcal{D}^+/[y^+, u^+]_{\boldsymbol{S}}$;
3) input–output decoupling zero at infinity (i.o.d.z. at ∞): $\mathcal{T}^+/(\mathcal{T}^+ \cap [y^+, u^+]_{\boldsymbol{S}})$;
4) hidden mode at infinity: $\mathcal{D}^+/(\Phi^+ \cap [y^+, u^+]_{\boldsymbol{S}})$;
5) invariant zero at infinity: $\mathcal{D}^+/[y^+]_{\boldsymbol{S}}$;
6) transmission (trans.) zero at infinity: $(\Phi^+ \cap [y^+, u^+]_{\boldsymbol{S}})/(\Phi^+ \cap [y^+]_{\boldsymbol{S}})$;
7) system pole at infinity: $\boldsymbol{D}^+/[u^+]_{\boldsymbol{S}}$;
8) transmission pole at infinity: $(\Phi^+ \cap [y^+, u^+]_{\boldsymbol{S}})/[u^+]_{\boldsymbol{S}}$;
9) controllable (cont.) pole at infinity: $\Phi^+/[u^+]_{\boldsymbol{S}}$;
10) observable (obs.) pole at infinity: $[y^+, u^+]_{\boldsymbol{S}}/[u^+]_{\boldsymbol{S}}$.

In addition, the order of the blocking zero at infinity of $\mathcal{D}^*$ is defined as that of the transfer matrix $G(s)$ (when it exists; see Definition 2).

Applying the same rationale as that in [2], one obtains the following result.

Proposition 4: The structural indexes of the transmission zero (respectively, pole) at infinity of $\mathcal{D}^*$ are the nonnegative (respectively, the opposite of the nonpositive) structural indexes of the transfer matrix $G(s)$ at infinity.

The following definitions of properness is an extension of definitions given in [25] and [11].

Definition 8: A linear newborn dynamics $\mathcal{D}^*$ is said to be internally (respectively, transfer-) proper iff the degree of its system (respectively, transmission) pole at infinity is zero.

From the rationale used in [2] one obtains the following theorem.

Theorem 2: The following properties hold, where $\delta(\cdot)$ denotes the degree of the module in parentheses and where ρ is the rank of $G(s)$ over $\boldsymbol{F}$ (or $\boldsymbol{L}$).

1) δ(hidden mode at ∞) = δ(i.d.z. at ∞) + δ(o.d.z. at ∞) − δ(i.o.d.z. at ∞).
2) δ(system pole at ∞)
$$= \begin{cases} = \delta(\text{i.d.z. at } \infty) + \delta(\text{cont. pole at } \infty)\\ = \delta(\text{o.d.z. at } \infty) + \delta(\text{obs. pole at } \infty)\\ = \delta(\text{trans. pole at } \infty) + \delta(\text{hidden mode at } \infty).\end{cases}$$
3) δ(trans. zero at ∞) + δ(i.o.d.z. at ∞) $\le$ δ(invariant zero at ∞).
4) If $\rho = p$, i.e., G is right-invertible, then δ(trans. zero at ∞) + δ(i.d.z. at ∞) $\le$ δ(invariant zero at ∞).
5) If $\rho = m$, i.e., G is left-invertible, then δ(trans. zero at ∞) + δ(o.d.z. at ∞) $\le$ δ(invariant zero at ∞).
6) If $\rho = m = p$, i.e., G is square and invertible, then δ(trans. zero at ∞) + δ(hidden mode at ∞) = δ(invariant zero at ∞).

B. Computations

In this section, we show how the structural indexes, orders and degrees of the various poles and zeros at infinity, can be computed in practice in the case of a PMD with time-varying coefficients of the form (7). First, write it in the form (2), and let $[A(\sigma), B(\sigma)]$ be a left-coprime factorization of $S^T(1/\sigma)$ over $\boldsymbol{S}$. Write

$$B(\sigma) = \begin{bmatrix} D^+(\sigma) & -N^+(\sigma) & Z^+(\sigma)\\ Q^+(\sigma) & W^+(\sigma) & Y^+(\sigma)\end{bmatrix} \tag{13}$$

according to the sizes in (7). The $\boldsymbol{S}$-module $\mathcal{D}^+$ is defined by (10)

1) The structural indexes of the ***i.d.z. at infinity*** are those of $B(\sigma)$.

Calculating the Smith form of $B(\sigma)$ in Example 1, $\mathcal{D}^*$ is found to have an i.d.z. at infinity with degree 1 (and order 1). The physical meaning of this is the same as the one pointed out by Verghese [27] in the case $\boldsymbol{K} = \mathbb{R}$; assume that the system is formed at some initial time t_0, due to a failure or a switch. Then, if the initial condition of ξ_1 is nonzero, an impulsive behavior occurs in the second row of the equations of $\mathcal{D}^*$ at time t_0^+, and cannot be eliminated using a nondistributional input; if u is a nondistributional input, then ξ_2

[1] They are not the same as those introduced in [10] (and used by several authors); see the concluding remarks.

is a distributional signal, and more specifically it is the *first order* derivative (in the sense of distributions) of a discontinuous function.

In the general case, with a nondistributional input, several partial states or outputs $v_{j_1}, \cdots, v_{j_k}$ are distributional ones, where every v_{j_l}, $1 \leq l \leq k$, is the derivative of order ω_1 of a discontinuous function. Then, the i.d.z. at infinity is of order max$\{\omega_1, \cdots, \omega_k\}$ and its degree is $\omega_1 + \cdots + \omega_k$ if the v_{j_l}, $1 \leq l \leq k$, are linearly independent in the suitable sense.

2) Similarly, the o.d.z., invariant zero and system pole at infinity are, respectively, characterized by the matrices $B_{odz}(\sigma)$, $B_{iz}(\sigma)$, $B_{sp}(\sigma)$ defined by

$$B_{odz}(\sigma) := \begin{bmatrix} D^+(\sigma) \\ Q^+(\sigma) \end{bmatrix}, \qquad B_{iz}(\sigma) := \begin{bmatrix} D^+(\sigma) & -N^+(\sigma) \\ Q^+(\sigma) & W^+(\sigma) \end{bmatrix}$$

$$B_{sp}(\sigma) := \begin{bmatrix} D^+(\sigma) & Z^+(\sigma) \\ Q^+(\sigma) & Y^+(\sigma) \end{bmatrix}.$$

3) The structural indexes of the *transmission zero at infinity* and of the *transmission pole at infinity* can be computed by applying Proposition 7.

4) The transfer matrix of $\mathcal{D}^*$ in Example 1 is (over $\boldsymbol{L}$) $G(s) = t^2\sigma^{-1}$. As a result, $\mathcal{D}^*$ is found to have a transmission pole at infinity with degree 1 (and order 1). The physical meaning of this is that for expressing y in function of u, one must differentiate u one time. $\mathcal{D}^*$ in Example 2 (respectively, three) has a transmission pole (respectively, zero) at infinity with indexes $\{k\}$, order k, and degree k.

The computation of the other infinite poles and zeros (see Definition 7) can be made using the same rules as those detailed in [2] for the finite ones (see also [5]).

V. Concluding Remarks

Definitions and properties of poles and zeros at infinity have been fully extended in this paper to the case of linear varying time-continuous systems. Our approach unifies all existing ones (in particular, it is shown in [6], using the so-called "normalized form" [27], [13] as well as minors of matrices and their valuations [18], that our definitions are consistent with those given in [1] in the time-invariant case). One of the advantages of defining a pole or a zero at infinity as being a module is that its whole structure is then captured (this has been already mentioned in [10], where infinite zero and pole modules associated with a transfer matrix are defined and studied). In our approach, the system is considered in an intrinsic manner instead of through one of its representations. Such a point of view has applications such as the choice of suitable input variables for obtaining an internally proper linear "newborn dynamics" [3].

The extension to the case of invariant discrete-time systems is obvious (the derivation s has only to be replaced by the forward shift operator q). The case of varying discrete-time systems is much more complicated because nonintegral rings must be used [12].

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