An intrinsic algebraic setting for poles and zeros of linear time-varying systems

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In this paper, poles and zeros are defined for linear time-varying systems using suitable ground field extensions. The definitions of the system poles, transmission poles, invariant zeros, hidden modes, etc, are given in an intrinsic module-based framework and are consistent in the sense that the poles are connected to the stability of the system and the zeros to the zeroing of the output for non zero inputs. In particular, it is proved that the necessary and sufficient condition for a continuous-time system to be exponentially stable is similar to the well-known condition in the time-invariant case.

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1. Introduction

The concept of poles of a system has received special attention as a means to evaluate stability. Although this relation is clear and well-known for linear time-invariant (LTI) systems, the time-varying case is much more difficult since the frozen-time (pointwise-in-time) eigenanalysis may not contain information about stability. Several definitions of poles of a linear time-varying (LTV) system have been proposed. In [17], Lyapunov transformations were used to obtain an upper triangular form of the state matrix, from which the notion of pole set of the state matrix was defined. In other approaches, factorizations of the differential operator associated with the system are used to evaluate stability ([12,20] and related references). Recent investigations have been carried on the polynomials over skew fields. In [18] it is shown that a skew polynomial can be written as a product of elementary factors $(\partial - \gamma)^{s}$ after a well-chosen (large enough) field extension while in [14] and related references the so-called Wedderburn polynomials (W-polynomials) are studied as the class of skew polynomials which factorize in distinct first order elementary factors. In the work presented here, independent first order factors are found for a given skew polynomial $P(X)$ using a suitable field extension. They form what is called in the sequel a fundamental set of those factors and it is shown that this is related to a fundamental set of solutions of the differential equation associated with the differential operator $P(\partial)$, $\partial = d/dr$ deduced from the skew polynomial $P(X)$. This allows us to give intrinsic definitions of the poles and zeros of LTV systems in the algebraic framework initiated by Malgrange [16] and popularized in systems theory by Fliess (see [6] and related references) in which an LTV system is modeled by a finitely presented module over a ring of linear differential operators.

The paper is organized as follows: Section 2 introduces the mathematical background, especially facts about field extensions, skew polynomials and differential operators. The different types of differential equations which may characterize an LTV system are also recalled. In Section 3 it is shown how first order factors can be obtained for a given skew polynomial using a well-chosen field extension. The latter field extension is shown to be a Picard–Vessiot one in Section 4 in which a fundamental set of solutions and independent roots are computed from the first order factors found in Section 3. The independent roots are not unique and their classes of conjugacy are studied in Section 5. The poles of an LTV system are introduced in Section 6 in a module-based formulation and the relation to stability is explained. In the same manner, modules for the other kinds of poles and zeros are introduced in Section 7. A physical example is presented in Section 8. The extension of this approach to the nonlinear case is discussed in Section 9, while Section 10 is devoted to concluding remarks.

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2. Notations and mathematical background

2.1. Skew polynomials, differential fields and operators

In everything that follows, all fields are commutative and of characteristic zero. Let $K = \mathbb{C}(t)$ be the (commutative) field of formal Laurent series with coefficients in $\mathbb{C}$; the field of complex numbers. An element $a$ of this field is of the form $a = \sum_{\ell=0}^{\infty} a_\ell t^\ell$, $a_\ell \in \mathbb{C}$. A field $L$ is a field extension of $K$ (written $L/K$) if $K \subseteq L$ is a subfield. Field extensions provide splitting fields for given polynomials, i.e., fields over which those polynomials split or, equivalently, are fully decomposable into products of first order factors. A Galois extension $L/K$ is an algebraic field extension for which every irreducible polynomial in the ring $K[x]$ of polynomials in $x$ with coefficients in $K$ admitting a zero in $L$ splits in $L$. Every Galois extension of $K$ over $\overline{\mathbb{C}}$ is of the form $K_m = \mathbb{C}(z)$, $z_m = t$, $z = t^{1/m}$, $m \in \mathbb{N}$ [18]. Moreover, $K_m$ has degree $m$ over $K$ and the union $\mathbb{K} = \bigcup_m K_m$ is an algebraic closure of $K$ (note that $\mathbb{K} \subseteq K_m$ for $m \leq n$).

Let $K$ be a differential field equipped with the usual derivation. Let $K[[\delta]]$ be the ring of left differential operators in $\delta = \partial/\partial t$ with coefficients in $K$, i.e., of all elements of the form $P(\delta) = \sum_{i=0}^{\infty} a_i \delta^{i-1}$, $a_i \in K$. It is equipped with the commutation rule $[a, b] = ab - ba$ that is the Leibniz rule of derivation of a product. Let $K[x; \delta]$ is the ring of skew polynomials with indeterminate $X$ and coefficients in $K$. The multiplication in $K[x; \delta]$ obeys to the commutation rule $Xa = aX + \partial a/\partial t$. One can associate with the skew polynomial ring $K[x; \delta]$ the ring of differential operators $K[[\delta]]$ by associating to the indeterminate $X$ the derivative operator $\delta$. The two rings are isomorphic, thus are identified. The following evaluation theory of skew polynomials was introduced in [13]: Let $K[x; \delta] / P(x) = \sum_{i=0}^{n} a_i X^i$ be a skew polynomial. The evaluation of $P$ in $a \in \mathbb{K}$, denoted by $P(a)$ is defined as follows: $P(a) = \sum_{i=0}^{\infty} a_i N_i(a)$ where $N_i$ is defined inductively by $N_0(a) = 1$, $N_i(a) = N_{i-1}(a) + \sum_{i=0}^{i} \frac{a}{\pi} (N_{i-1}(a))$. For $P \in K[x; \delta] / a \in \mathbb{K}$, $P(a) = 0$ if and only if $X - a$ is a right factor of $P$ and, in this case, we say that $a$ is a (right) root of $P$. For any nonzero $c \in \mathbb{K}$, the conjugate of $a$ by $c$ denoted by $a^c$ is $a^c = a + \frac{c}{\pi} a^{-1}$, $\Delta(a) = \{a^c, c \in \mathbb{K}, c \neq 0\}$ is called the conjugacy class of $a$. Let $G, P \in K[x; \delta]$. A product formula

$$(PG)(a) = 0, \quad \text{if } G(a) = 0 \quad \text{or } P(a^G)G(a), \quad \text{if } G(a) \neq 0 \quad (1)$$

as well as the existence of a least common multiple denoted by $[P, G]$ can be established for skew polynomials. $[P, G]$ is a monic polynomial such that $[K(X; \delta)] \cap [K(X; \delta) / G] = [K(X; \delta) / P, G]$. There are several ways of computing the least common multiple which lead thus to several factorizations of $[P, G]$. Relation (1) clarifies the calculations in the case of first order factors: let $P \in K[x; \delta]$ and $\alpha \in \mathbb{K}$. Then

$$(P, X - \alpha) = \beta, \quad \text{if } P(\alpha) = 0 \quad \text{or } (X - \alpha)^{P(\alpha)}P, \quad \text{if } P(\alpha) \neq 0 \quad (2)$$

Let $\Delta = \{\gamma_1, \ldots, \gamma_n\} \subseteq \mathbb{K}$. The minimal polynomial of $\Delta$, denoted by $P_{\Delta}$, is the monic generator of the left ideal $[P(X; \delta) / \Delta, \Delta] = (0)$; $P_{\Delta} = (\delta - \gamma_i, \delta = 1, \ldots, n)$. $\Delta$ is said $P$-independent if the degree of $P_{\Delta}$ is $n$. Relation (2) can be applied to construct the minimal polynomial in the generic case in a recursive manner.

2.2. LTV systems

In the approach adopted by Fliess (see [6] and related references) an LTV system is a finitely presented module $M$ over a noncommutative ring $K[\delta]$. Assuming that this system $M$ is equipped with an input $u = (u_i)_{i=1, \ldots, m}$, i.e., a set of variables such that $[u_i, \mathbb{R}]$ is the $\mathbb{R}$-module generated by the entries of $u$, is free of rank $m$ and $M/[u_i\mathbb{R}]$ is torsion, stability can be studied on the so-called autonomous part of the system, obtained by canceling the input variables; algebraically, we obtain in this way the torison module $T = M/[u_i\mathbb{R}]$ as $R$ is simple, $T$ is cyclic and can be characterized by a state-space representation $\phi^T = Ax, A \in \mathbb{K}^{n \times n}$ or, equivalently, by a scalar differential equation

$$P(\partial) y = 0, \quad P(\partial) = \partial^n + \sum_{i=1}^{n} a_i \partial^{n-i} \quad (3)$$

where $y$ is a generator of $T$.

3. Polynomial factorization

In [18] it is shown how a suitable field extension $K_m / K$ can be constructed in order to find factors of a skew polynomial belonging to $K_m \langle x; \delta \rangle$. This procedure is shown on

Example 1. Consider (3) with $P(\partial) = \partial^3 + (\theta^{-1} - 2\alpha)\partial - t^3 - \theta^{-1} - 2\alpha^2$, $\alpha \in \mathbb{K}$. The change of variable $\xi = t^{\theta} \partial$ transforms the initial equation into a differential equation with coefficients in the ring of formal series in $t^{3/2}, Q_2 = [[t^{3/2}]]$. Setting $P(\partial) = L(\xi)$, $L(0) = 0$ and $L(\xi) = t^{-3/2}(\xi)$, $Q(\xi) = \xi^2 + (\theta^{-1} - 2\alpha^2) - 1 - \theta^{-1} - \alpha^2$ where the commutation rule for the new variable $\xi$ is deduced from the initial one (introduced in Section 2.1), i.e., $\xi = a(t) + \theta^{-1} \partial$, $a \in \mathbb{K}$. Therefore, the polynomial whose coefficients are the canonical images of those of $Q(\xi)$ in $\mathbb{Q}_0 / \pi \mathbb{Q}_0, \pi = t^{1/2}$, is $Q(\xi) = \xi^2 - 1$ which trivially factorizes over $C: Q(\xi) = F_1 F_2$, with $F_1 = \xi + 1$ and $F_2 = \xi - 1$. Using Hensel’s Lemma [18], a unique factorization $Q(\xi) = Q_1(\xi) Q_2(\xi)$ such that $Q_1(\xi) = F_1$, $i \in \{1, 2\}$ can be obtained from the factors $F_1, F_2$ via an iterative process in which terms of the form $\pi^r T_1, \pi^r S_2$ are added to the factorials $F_1$ and $F_2$ in such a regular way that $f_0(T_1) = f_0(f_2)$ and $d_0(S_2) = d_0(f_2)$, where $d_0(.)$ denotes the degree of a skew polynomial. For this example, $T_1, S_2 \in C$ and the factors can be found in one step: $Q(\xi) = \xi - 1 - \theta^{-1} - \alpha^2$, $Q_2(\xi) = \xi - 1 - \theta^{-1} - \alpha^2$. Obviously, $Q_2(\xi) = F_2$, $i \in \{1, 2\}$ and one can easily check using the new commutation rule in $\xi$ above that $Q(\xi) = Q_1(\xi) Q_2(\xi)$. These facts enable one to find a factorization of $P(\partial)$. Here, $P(\partial) = t^{-3/2} Q_2(\theta^{3/2} \partial) Q_2(\theta^{3/2} \partial) = t^{3} \left( t^2 \partial - 1 - \theta^{-1} - \alpha^2 \right)^2 \left( t^2 \partial + 1 - \theta^{-1} - \alpha^2 \right)^2 = t^3 \left( t^3 \left( \theta^{-1} - \alpha \right) - \theta^{-1} - \alpha \right)$. This leads to a decomposition of $P$ with two first order factors: $P(\partial) = (\partial - a_2) (\partial - a_1)$, where $a_1(t) = -t^3 - \theta^{-1} - \alpha$ and $a_2(t) = t^3 - \theta^{-1} - \alpha$. Generally, the polynomial $P(\partial)$ in (3) can be decomposed into elementary factors, namely

$$P(\partial) = (\partial - a_2)^{d_2} \cdots (\partial - a_1)^{d_1} \quad (4)$$

over a suitable extended field $[a_i \in K_m$ for some $m$. The elementary factors are thus of order 1 (di = 1) or greater.

4. Independent roots and solutions, Picard–Vessiot extensions

The solution space of (3) has a dimension $n$ over $C$. A basis of this space is called a fundamental set of solutions of (3). A set $\{y_1, \ldots, y_n\}$ of particular solutions of (3) form such a basis if and only if the Wronskian matrix $W$ in (5) is nonsingular. A
A fundamental set of roots of $P$ can be constructed starting from the factors of the polynomial $P$. The factorization $(4)$ is obtained over a Galois extension $K_m$ for $P$. As the derivative $\delta$ has a unique extension from $K$ to $K_m$, it can be shown that $K_m$ is also a Picard–Vessiot field of the differential equation $(3)$ and thus has $n$ $C$-linearly independent solutions over $K_m$ and $P$ has $n$ $P$-independent roots over that field. A fundamental set of roots can be found from the elementary factors of $(4)$ in an iterative way: first, $\gamma_1 = a_1$. Next, $\gamma_2$ is solution of the equation $y_2^{\gamma_2 - \gamma_1} = a_i$ with $i = 1$ if $d_1 > 1$ or $i = 2$ if $d_1 = 1$ which leads to the Riccati equation in $y_2$: $\frac{dy_2}{y_2^2} = (a_1 + \gamma_1)y_2 - \frac{\gamma_2}{y_2} + \gamma_2 a_1, i = 1$ or 2. The latter Riccati equation has the particular solution $y_2 = \gamma_1$ and is thus solvable by reduction to a linear first order equation. Doing so in the same manner for the rest of the factors $a_i$ (multiply or not), one obtains a fundamental set of roots of $P$, $\Delta = \{ \gamma_1, \ldots, \gamma_n \}$. Notice that the $\gamma_i$’s may not necessarily all belong to $K_m$ but to a larger field $K_n$ which is still a Picard–Vessiot extension of $K$ but with the additional property that it allows the computation of a fundamental set of roots. It follows that:

\[ \hat{K}_m[\delta]P = \bigcap_{i} \hat{K}_m[\delta](\partial - \gamma_i), \quad i = 1, \ldots, n. \] (6)

As well, the factors $(\partial - \gamma_i)$ are of order 1 and thus coprime, $(6)$ is a complete direct decomposition of $\hat{K}_m[\delta]P$ [4]. For Example 1, a root is $\gamma_1 = a_1 = -t^{-\frac{1}{2}} + \frac{1}{2} t + \alpha$. As a second root of a fundamental set can be found as a solution of the equation $y_2^{\gamma_2 - \gamma_1} = a_2$: $y_2 = t^{-\frac{1}{2}} + \frac{1}{2} t + \alpha$.

A fundamental set of solutions of $(3)$ can be found from a fundamental set of right roots of $P$: $\gamma_i = 1, \ldots, n$ as introduced in Section 2, by solving the elementary equations

\[ (\partial - \gamma_i)y_i = 0 \iff \frac{dy_i}{y_i} = y_i, \quad i = 1, \ldots, n. \] (7)

Thus $y_i(t) = e^{\int \gamma_i(t)dt}, i = 1, \ldots, n$, which for Example 1 leads to $y_{1,2} = e^{\int (\gamma_1(t) - \gamma_2(t))dt}$ which are $C$-independent. Indeed, $W(y_1, y_2) = \begin{bmatrix} y_1 & \frac{dy_1}{y_1} \\ y_2 & \frac{dy_2}{y_2} \end{bmatrix}$ and $\det(W(y_1, y_2)) = 2y_2(t)y_1(t)t^{-\frac{1}{2}} \neq 0$ for $t > 0$.

It is thus always possible to find a field extension over which a fundamental set of roots of a given polynomial $P(\delta)$ can be provided. This construction is done without explicit calculation of the solutions of $(3)$:

Proposition 3. Let $P(\delta) \in K[\delta]$ and let $\hat{K}_m$ be a Picard–Vessiot extension for $(3)$ as computed above. Then:

(i) a fundamental set of roots $\{ \gamma_1, \ldots, \gamma_n \}$ of $P$ can be found in $\hat{K}_m$.

(ii) the set $\{ y_i, i = 1, \ldots, n \}$ where $y_i(t) = e^{\int \gamma_i(t)dt}$ is a fundamental set of solutions of $(3)$. 

5. Conjugacy classes of roots

Consider an autonomous system defined over the Puisseux-type field $K^\infty = \bigcup_{m \geq 1} C(t^{1/m})$ and let $R^\infty = K^\infty[\delta]$; the above system is, up to a $R^\infty$-isomorphism, the torsion $R^\infty$-module $T^\infty$ defined by $(3)$.

Definition 2. Let $T^\infty$ be a torsion $R^\infty$-module, $T^\infty \cong R^\infty/R^\infty P(\delta)$ and let $\{ \gamma_1, \ldots, \gamma_n \}$ be a fundamental set of roots of $P$. The module $T^\infty$ or the autonomous system associated to $T^\infty$ is called regular near $+\infty$ if there exists $t_0 \in \mathbb{R}$ such that all functions $\gamma_i: t \rightarrow \gamma_i(t)$ are analytic on $(t_0, +\infty)$. 

Definition 1. Let $P \in K[\delta]$ be a polynomial of degree $n$. A fundamental set of roots of $P$ is a set $\Delta = \{ \gamma_1, \ldots, \gamma_n \}$ of right roots of $P$ such that $P = P_{\Delta}$. 

A fundamental character can be given for the roots of the polynomial $P$. Note first that in a factorization of type $(4)$, only $a_1$ is a right root (zero) of $P$ in the sense of the definition given in Section 2.1 for skew polynomials. The resulting set will be used in what follows. It is a to the conjugacy class of $0 \in k$ of a more general property established for $W$-polynomials in [14] and related references.
For $T^\infty \cong \mathbb{R}^\infty /\mathbb{R}^\infty P(\hat{\beta})$ to be regular near $+\infty$ it is sufficient that the iterative procedure described in Example 1 converges in a finite number of steps. Also, from (7) it is obvious that if an autonomous system is regular near $+\infty$, for any fundamental set of solutions $(\gamma_i)_{i=1,\ldots,n}$, there exists $t_0 \in \mathbb{R}$ such that each $\gamma_i$ ($i = 1, \ldots, n$) is analytic on $(t_0, +\infty)$.

Extending the ring of scalars to $\mathbb{K}^\infty = \mathbb{K}[\hat{\beta}]$ where $\mathbb{K}^\infty = \mathbb{K}[[\gamma_1, \ldots, \gamma_n]]$ is the field of rational functions of a fundamental set of roots $\gamma_i$ of $P$ with coefficients in $\mathbb{K}^\infty$, the decomposition of type (6) leads to the following direct sum of modules

$$\tilde{T}^\infty \cong \tilde{\mathbb{R}}^\infty /\tilde{\mathbb{R}}^\infty P(\hat{\beta}) \cong \oplus_i \tilde{\mathbb{R}}^\infty /\tilde{\mathbb{R}}^\infty (\hat{\beta} - \gamma_i)$$

(8)

which is a complete direct decomposition of $T^\infty \cong \mathbb{R}^\infty \otimes_{\mathbb{R}^\infty} T^\infty$. For Example 1, one obtains $T^\infty \cong \mathbb{R}^\infty \otimes_{\mathbb{R}^\infty} T^\infty (\hat{\beta} - \gamma_1^2 - \frac{1}{2}\gamma_1^{-1} - \alpha) \oplus \mathbb{R}^\infty /\mathbb{R}^\infty (\hat{\beta} - \gamma_1^2 - \frac{1}{2}\gamma_1^{-1} - \alpha).

In the module-based framework, (3) is not the unique differential equation which defines the autonomous system. Indeed, if $T^\infty \cong T^\infty$ is a module over $\mathbb{R}^\infty$ defined by the equation

$$P(\hat{\beta})\tilde{\mathbb{R}}^\infty = 0$$

then (3) and (9) define the same autonomous system. We say also that $P(\hat{\beta})$ and $P(\hat{\beta})$ are similar [4], written $P(\hat{\beta}) \sim P(\hat{\beta})$, when $\mathbb{R}^\infty /\mathbb{R}^\infty P(\hat{\beta}) \cong \mathbb{R}^\infty /\mathbb{R}^\infty P(\hat{\beta})$. Each $\gamma_i$ is conjugated with some root $\gamma_j$ of $P(\hat{\beta})$ over $\mathbb{K}^\infty$. The link between the roots of the similar skew polynomials has been studied in [15] where it was shown that there exists $Q \in \mathbb{R}^\infty$ such that $\tilde{\gamma}_i = Q(\gamma_j)$ and $V(Q) \cap V(P) = \emptyset$ where for $F \in \mathbb{R}^\infty$, $V(F) = \{ x \in \mathbb{K}^\infty, F(x) = 0 \}$. The conjugacy class of any element $\gamma_i$ of a fundamental set of roots is thus of the form

$$\Delta_{\mathbb{K}^\infty}(\gamma_i) = \left\{ \tilde{\gamma}_i = \gamma_i + \frac{dc}{dt}t^{-c}, 0 \neq c = Q(\gamma_i), Q \in \mathbb{R}^\infty \right\}.$$  

(10)

It can be shown that in the situation above where $c$ is the evaluation of a polynomial in a given $\gamma_i$, $\frac{dc}{dt} \rightarrow 0$ as $t \rightarrow \infty$. Thus, the conjugacy class (10) is infinite but such that as $t \rightarrow \infty \Rightarrow \gamma_i - \tilde{\gamma}_i \rightarrow 0, \forall \tilde{\gamma}_i \in \Delta_{\mathbb{K}^\infty}(\gamma_i)$.

6. Poles and stability of an autonomous LTV system

The stability of a system can be evaluated from the stability of the autonomous system. It can be defined and studied in the intrinsic module framework: consider an autonomous LTV system given by the $\mathbb{R}^\infty$-torsion module $T^\infty \cong \mathbb{R}^\infty /\mathbb{R}^\infty P(\hat{\beta})$ and let $\gamma_i, i = 1, \ldots, n$ be a fundamental set of roots of the polynomial $P(\beta)$.

(i) The poles of $\Sigma$ are the conjugacy classes $\Delta_{\mathbb{K}^\infty}(\gamma_i), i = 1, \ldots, n$.

(ii) A set $\{\tilde{\gamma}_i, \ldots, \tilde{\gamma}_n\}$ of representatives of these poles (i.e., $\tilde{\gamma}_i \in \Delta_{\mathbb{K}^\infty}(\gamma_i)$) is called fundamental if the $\tilde{\gamma}_i$'s are $P$-independent.

(iii) The module of poles of $\Sigma$ is $\tilde{T}^\infty$ defined by (8).

Notice that the poles of $\Sigma$ only depend on $\tilde{T}^\infty$ (not on $P(\hat{\beta})$). Thus, this definition of poles is intrinsic. For Example 1, a fundamental set of representatives of poles is thus $\gamma_{1,2} = \frac{-1}{2} + \frac{1}{2}t^{-1} + \alpha$. As a fundamental set of solutions to (3) is given by the elementary equations (7), stability can be directly evaluated in terms of the poles of the system:

**Theorem 1.** The regular near $+\infty$ autonomous system $\Sigma$ defined by the $\mathbb{R}^\infty$-torsion module $T^\infty$ is exponentially stable if and only if

$$\lim sup_{t \rightarrow +\infty} Re[\gamma(t)] < 0, \quad i = 1, \ldots, n$$

(12)

where $\gamma_1, \ldots, \gamma_n$ is any fundamental set of representatives of poles of $\Sigma$. This system is exponentially unstable if, and only if there exists an index $i \in \{1, \ldots, n\}$ and $\gamma \in \Delta_{\mathbb{K}^\infty}(\gamma_i)$ such that

$$\lim inf_{t \rightarrow +\infty} Re[\gamma(t)] > 0.$$  

(13)

**Proof.** Let $\gamma_1, \ldots, \gamma_n$ be a fundamental set of representatives of poles of the autonomous system. A definition equation of $T^\infty$ is (3), where $\gamma_i$ is the minimal polynomial of $\gamma_1, \ldots, \gamma_n$. A fundamental set of solutions of (3) is provided by the solutions $\gamma_i$ of the elementary equations (7). If (12) holds, then $Re[\gamma(t)] \leq -\beta_i$ for $t$ large enough, for some $\beta_i > 0, \forall i = 1, \ldots, n$. It follows that for $i = 1, \ldots, n$ and a time $t_1$ there exist finite, positive constants $\alpha_i$ and $\beta_i$, such that

$$|e^{\beta_i t}| \leq \alpha_i e^{-\beta_i (t-t_1)}, \quad \forall t \geq t_1 > t_0 \Leftrightarrow |\gamma_i| \leq \alpha_i e^{-\beta_i (t-t_1)}$$

$$t_1 > t_0$$

(14)

from which one concludes that any solution $Y(t)$ satisfies (11).

Let now $\gamma_1, \ldots, \gamma_n$ be another set of poles computed for $\tilde{T}^\infty \cong \tilde{T}^\infty$. Then, as shown in Section 5, $\gamma_i$ is conjugated to $\gamma_i$ by equation $\gamma_i = \gamma_i + \frac{dc}{dt}t^{-c}(10)$ and $\gamma_i - \tilde{\gamma}_i \rightarrow 0$ as $t \rightarrow \infty$. It follows that, if $\gamma_1, \ldots, \gamma_n$ satisfy condition (14), then the same condition is satisfied by any set $\gamma_1, \ldots, \gamma_n$ of poles from the conjugacy class (10) as $t \rightarrow \infty$ and thus the solutions $\gamma_i = e^{\beta t}\tilde{\gamma}_i$ also satisfy (11). Conversely, if the system is exponentially stable, then for any solution $\gamma_i$ of a fundamental set of solutions of (3) associated to $\tilde{T}^\infty$ there exist $\alpha > 0, \beta_i > 0$ to satisfy (14) where $\gamma_i = \frac{dc}{dt}t^{-c}$ from which (12) follows.

Conditions for asymptotic stability can be obtained in the same manner. Due to the space limitation, their statement is left to the reader. For the system in Example 1, condition (12) is satisfied if and only if $\alpha < 0$ and condition (13) is satisfied if and only if $\alpha > 0$. The system is thus exponentially stable if and only if $\alpha < 0$, exponentially unstable if and only if $\alpha > 0$ and when $\alpha = 0$ the system is neither exponentially stable nor exponentially unstable. It can be shown that in the latter case the system is however unstable. Indeed, the solutions of (3) with $P(\hat{\beta})$ of Example 1 are $Y(t) = \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t} + \alpha_3 e^{\beta_3 t} + \alpha_4 e^{\beta_4 t}$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$. Thus, as $t \rightarrow +\infty$, $|Y(t)| \rightarrow 0$ exponentially if $\alpha < 0$ and $|Y(t)| \rightarrow +\infty$ exponentially if $\alpha > 0$. Notice that in case $\alpha = 0$, also $|Y(t)| \rightarrow +\infty$ but not exponentially.

7. Modules of poles and zeros of an LTV control system

The poles and zeros of the LTV systems have been defined and studied in the module-based framework in [1] where to each kind of pole or zero it was associated a module deduced from the one which defines the system. This approach has been used in [2] to define the structure at infinity of an LTV system in an
intrinsic way. Next, these entities of modules of system poles and zeros have been proposed in [3] for the finite structure of LTV systems also. This framework can now be fully validated by considering the field extensions and constructions described above which allows one to define the poles of a given LTV system in the general case where they do not belong to the initial field of definition of the system $\mathbb{K}$. Indeed, let $M_0 \cong M^\infty/[u]_{M^\infty}$, where $M^\infty = \hat{\mathbb{R}} \otimes_{\mathbb{R}} M^\infty$ ($\otimes$ denotes the tensor product) be the module of poles of the LTV system defined by the $\mathbb{R}^\infty$-module $M^\infty$. Let $B(\delta)$ be a definition matrix of $M_0$, $B(\delta)w = \delta w$ and consider its Jacobson–Teichmüller normal form [4] over $\hat{\mathbb{K}}$: there exist unimodular matrices $U, V$ with entries in $\hat{\mathbb{R}}$ such that $UBV^{-1} = \text{diag}(1, \ldots, 1, P_m(\delta))$. $P_m$ is defined up to similarity and $P_m \cong P$, where $P$ is the polynomial in (3) of $M_0$. A set of poles of the LTV system defined by $M^\infty$ is thus a fundamental set of roots of $P_m$. Let $\delta x = Ax$ be a state-space representation of $M_0$, $A \in (\mathbb{K}^\infty)^{n \times n}$. The same kind of characterization is obtained if the similarity of matrices [4] defining the same module is considered: $\lambda I - A \sim \text{diag}(\lambda_1, \ldots, \lambda_m, \lambda_i(\partial))$ and the right eigenspectrum of $A$ is $\bigcup_{\lambda_i(\partial) \in \mathbb{C}} \varphi(f)$ where $\lambda_i(P) = [\lambda \neq \lambda_i \sim P]$ [15]. The family of the sets of poles of the LTV system defined by $M^\infty$ is thus the family of the right eigenspectrums of the state matrices of $M_0$.

The same construction can be made for the zeros if one considers a vector $z$ of output variables of an LTV system defined by the $\mathbb{R}^\infty$-module $M^\infty$. Let $y$ be a generator of $T(M^\infty/[z]_{M^\infty})$ where $T(.)$ denotes the torsion submodule. There exists $Z(\delta) \in \mathbb{K}[\delta]$ such that $Z(\delta) = 0$ and let $z_1, \ldots, z_j \in \mathbb{K}$ be a fundamental set of roots of $Z$. If $(\mathbb{K}^\infty)^{c \times c}$ with respect to controllability/controllability. The general Kalman decomposition can be obtained in the same way, with a similar interpretation.

**Case of transmission zeros:** They have the usual significance, i.e., they give the class of nonzero inputs for which a zero output is obtained as shown on the following example:

**Example 2.** Let $G(\delta) = \begin{bmatrix} 1 & (\delta + 1)^{-1} & (\delta + 1)^{-1}(\delta + 2) \\ 0 & 0 & (\delta + 1)^{-1} \end{bmatrix}$. A factorization of $G$ is provided by computing the left common multiple of $(\delta + 1)$ and $(\delta + 1)$: $P(\delta) = [\delta + 1, \delta + 1] = \partial^2 + \frac{2-2\sigma}{1-\gamma} \partial + \frac{1-\gamma}{1-\gamma}$. Then, $G(\delta) = D^{-1}N_2$, where $D = \begin{bmatrix} P(\delta) & 0 \\ 0 & \sigma + 1 \end{bmatrix}$, $N_2 = \begin{bmatrix} P(\delta) & \delta - \frac{2-2\sigma}{1-\gamma} & \frac{1-\gamma}{1-\gamma} \end{bmatrix}$ (\delta + 2)$. Notice that $D, N_2$ are also left-coprime (when it is not the case, such a pair must be constructed from $D, N_2$, i.e., a greatest left factor must be eliminated) and thus they are matrices of definition of the system modules of poles and respectively transmission zeros. It follows that a fundamental set of representatives of the transmission poles is $\{-1, -1\}$ and one for the transmission zeros is $\{\frac{1}{1-\gamma}\}$. The latter is unique since it consists of only one element. Therefore, there exist a nonzero $\mathbb{R}^\infty$-linear combination $\varphi$ of the inputs satisfying $(\delta - \frac{2-2\sigma}{1-\gamma}) = 0$ and an $\mathbb{R}^\infty$-linear combination $\varphi$ of the outputs, with nonzero coefficients, such that $\varphi = 0$ for the above inputs.

The same relations as in the LTI case [1] exist between the poles and zeros defined above. They are consequences of inclusion relations of the torsion modules to which they are associated. For any torsion $\mathbb{R}^\infty$-module $T$ of poles or zeros defined above, let $\Gamma(T) = \bigcup_{\varphi \in \mathbb{R}^\infty} \{\gamma_1, \ldots, \Gamma(T) \subseteq M_0 \}$ and $T_2 \subseteq T_1$. Then $\Gamma(T_1) = \Gamma(T_2) \cup \Gamma(T_1/T_2)$ where $\cup$ denotes the disjoint union.

**Proposition 4.** Let $T_2 \subseteq T_1$ be two finitely generated torsion modules over $\mathbb{R}^\infty$. Then $\Gamma(T_1) = \Gamma(T_2) \cap \Gamma(T_2/T_1)$ where $\cap$ denotes the disjoint union.

This leads to the usual relations between the sets of poles and zeros:

- $\Gamma(M_0) = \bigcap_{\varphi \in \mathbb{R}^\infty} \Gamma(M_0)$
- $\Gamma(M_0) = \bigcap_{\varphi \in \mathbb{R}^\infty} \Gamma(M_0)$

For example, $M_0 \subseteq M_0$ and $M_0 \cong M_0$. Using Proposition 4 with $T_1 = M_0$ and $T_2 = M_0$, one gets $\Gamma(M_0) = \bigcap_{\varphi \in \mathbb{R}^\infty} \Gamma(M_0)$.

**8. A physical example**

A bandwidth adaptation of linear filters is sometimes needed to fulfill the specifications of several classes of industrial applications. Such an example is treated in [21] where a time-varying bandwidth filter of the form

$$P(\delta)u_{\text{out}} = u_{\text{in}},$$

$$P(\delta) = \partial^2 + \left[2\xi \cos(\delta) - \frac{\sin(\delta)}{\omega_n(\delta)^{-1}}\right] \partial + (\cos(\delta))^2.$$  (15)
is used in the design of a missile autopilot. In order to cope with the requirements on the tracking performance and on the actuator rate limit, the control is filtered by a block of type (15) which is supposed to reduce the acceleration and the rate of the control in case of tracking of abrupt trajectories and, on the contrary, to have little influence on smooth trajectories which can be tracked without limiting the actuator. These two requirements cannot be achieved with a fixed-parameter filter and a well-chosen time dependence $t \mapsto \omega_n(t)$ in (15) of the bandwidth should be considered. Consider the autonomous part

$$P(\bar{\partial})y = 0$$

of (15). For $0 < \xi < 1$, the solutions of (16) can be easily found to be $y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$, $y_{1,2}(t) = e^{(1-\xi i \sqrt{1-\xi^2})t} / \omega_n(t)dt$ where $i^2 = -1$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. According to the elementary equations (7), to the fundamental set of solutions $\{y_1(t), y_2(t)\}$ above corresponds the fundamental set of roots $\{\gamma_1(t), \gamma_2(t)\}$ of $P$: $\gamma_{1,2} = -\xi \omega_0(t) \pm i \omega_0(t) \sqrt{1 - \xi^2}$. As the bandwidth variation is necessarily such that $\omega_0(t) \geq \omega_0 > 0$, $\forall t$, condition (12) of Theorem 1 is satisfied and the filter (15) is thus exponentially stable.

9. Extensions to the nonlinear case

A nonlinear system $\Sigma$ over a differential ground field $k$ is a finitely generated extension $L/k$. Denoting by $d_{L/k}$ the Kähler differential [10], the $L[\partial]$-module $\Omega_{L/k}$ spanned by the elements $d_a \Omega_k$, $a \in L$ is the tangent linear system $\Sigma_T$ associated with $\Sigma$ [8]. $\Sigma_T$ is an LTV system; its transfer matrix $G(\bar{\partial})$ can be calculated as in [7]. The above theory can be applied to $\Sigma_T$ with $k = K^\omega$, and $\Sigma$ is said to be exponentially stable if so is $\Sigma_T$ (in accordance with the Lyapunov theory). $G(\bar{\partial})$ can be called the transfer matrix of $\Sigma$ (as well as of $\Sigma_T$) and has connections with that calculated in [9].

10. Conclusion

Poles and zeros have been defined for LTV systems by extending the field over which the system is given. It has been shown that these definitions are consistent; as a matter of fact, the system poles are connected to the stability of the system and the invariant zeros to the zeroing of the output for non zero inputs, when the field extension is a Picard–Vessiot one for the system. Poles and zeros are not unique, as opposed to their conjugacy classes; this is consistent with the intrinsic module-based approach for LTV systems used in this paper.

References