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Structures in Concrete Categories

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Abstract

The monumental treatise "Éléments de mathématique" of N. Bourbaki is based on the notion of structure and on the theory of sets. On the other hand, the theory of categories is based on the notions of morphism and functor. An appropriate definition of a structure in a concrete category is given in this paper. This definition makes it possible to bridge the gap between Bourbaki's structural approach and the categorical language.

1 Introduction

The monumental treatise "Éléments de mathématique" of N. Bourbaki is based on the notion of structure and on the theory of sets [2]. On the other hand, the theory of categories is based on the notions of morphism and functor. The notion of "concrete category", as developed in [1], did not bridge the gap between the categorical approach and Bourbaki's structures. According to ([1], Remark 5.12), a structure in a concrete category $(\mathcal{C}, |.|, \mathcal{X})$, where $\mathcal{X}$ is the base category and $|.| : \mathcal{C} \to \mathcal{X}$ is the forgetful functor, is an object A of $\mathcal{C}$. If A, A' are two $\mathcal{C}$-isomorphic objects such that $|A| = |A'|$, these objects are not equal in general. Nevertheless, if $\mathcal{C}$ is, for example, the category **Top** of topological spaces and $\mathcal{X}$ is the category **Set** of sets, A and A' are two equal sets endowed with the same topology.

Definition 1 *A structure in $(\mathcal{C}, |.|, \mathcal{X})$ is an equivalence class of objects of $\mathcal{C}$ with respect to the following equivalence relation: two objects A and A' of $\mathcal{C}$ are equivalent if $|A| = |A'|$ and there is a $\mathcal{C}$-isomorphism $A \to A'$.*

Some consequences of this definition are studied the sequel.

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2 Structures

2.1 Isomorphisms of structures

Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two structures in $(\mathcal{C}, |.|, \mathcal{X})$. Assume that there exist representatives A_1 and A_2 of $\mathfrak{S}_1$ and $\mathfrak{S}_2$, respectively, which are isomorphic in $\mathcal{C}$. Then $|A_1|$ and $|A_2|$ are not necessarily equal, but are isomorphic in $\mathcal{X}$. If A_1' and A_2' are other representatives of $\mathfrak{S}_1$ and $\mathfrak{S}_2$, they are isomorphic in $\mathcal{C}$.

Definition 2 *The structures $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are called isomorphic if the above condition holds.*

This definition determines an equivalence relation on structures in $(\mathcal{C}, |.|, \mathcal{X})$.

2.2 Comparison of structures

Obviously, if $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two structures in $(\mathcal{C}, |.|, \mathcal{X})$, and A_1 and A_2 are representatives of $\mathfrak{S}_1$ and $\mathfrak{S}_2$, respectively, such that $|A_1| = |A_2|$ and there exists a $\mathcal{C}$-morphism $A_1 \to A_2$ such that $|A_1 \to A_2| = \mathrm{id}_{|A_1|}$, then for any representatives A_1' and A_2' of $\mathfrak{S}_1$ and $\mathfrak{S}_2$, respectively, there exists a $\mathcal{C}$-morphism $A_1' \to A_2'$ such that $|A_1 \to A_2| = \mathrm{id}_{|A_1'|}$.

Definition 3 *If the above condition holds, then the structure $\mathfrak{S}_1$ is called finer than $\mathfrak{S}_2$ (written $\mathfrak{S}_1 \geq \mathfrak{S}_2$) and $\mathfrak{S}_2$ is called coarser than $\mathfrak{S}_2$ (written $\mathfrak{S}_2 \leq \mathfrak{S}_1$).*

Lemma 4 *The relation $\geq$ on structures in $(\mathcal{C}, |.|, \mathcal{X})$ is an order relation.*

Proof. The reflexivity and the transitivity are obvious. Assume that $\mathfrak{S}_1 \geq \mathfrak{S}_2$ and $\mathfrak{S}_2 \geq \mathfrak{S}_1$. Let A_1, A_2 be representatives of $\mathfrak{S}_1, \mathfrak{S}_2$ respectively. There exist $\mathcal{C}$-morphisms $A_1 \to A_2$ and $A_2 \to A_1$ such that $|A_1 \to A_2| = \mathrm{id}_{|A_1|}$ and $|A_2 \to A_1| = \mathrm{id}_{|A_2|}$. Thus $A_1 \to A_2 \longrightarrow A_1$ and $A_2 \to A_1 \to A_2$ are $\mathcal{C}$-morphisms such that $|A_1 \to A_2 \longrightarrow A_1| = \mathrm{id}_{|A_1|}$ and $|A_2 \to A_1 \to A_2| = \mathrm{id}_{|A_2|}$. Since the forgetful functor is faithful, it follows that $A_1 \to A_2 \longrightarrow A_1 = \mathrm{id}_{A_1}$ and $A_2 \to A_1 \to A_2 = \mathrm{id}_{A_2}$. Therefore, $A_1 \to A_2$ and $A_2 \to A_1$ are isomorphisms, and each of them is the inverse of the other. As a result, $\mathfrak{S}_1 = \mathfrak{S}_2$. ■

Example 5 *A structure in the concrete category $(\mathbf{Top}, |.|, \mathbf{Set})$ is a topology. The structure $\mathfrak{S}_1$ is finer than $\mathfrak{S}_2$ if, and only if the topology $\mathfrak{S}_1$ is finer than the topology $\mathfrak{S}_2$.*

2.3 Initial structures and final structures

Let A, B be objects of $\mathcal{C}$. Recall that the relation "$f : |B| \to |A|$ is a $\mathcal{C}$-morphism" means that there exists a morphism $f : B \to A$ (necessarily unique) such that $|B \to A| = |B| \to |A|$ ([1], Remark 6.22).

Let $(f_i : A \to A_i)_{i \in I}$ be a source in $\mathcal{C}$, i.e. a family of $\mathcal{C}$-morphisms. This source is called initial if for each object B of $\mathcal{C}$, the relation

"$f : |B| \to |A|$ is a $\mathcal{C}$-morphism"

is equivalent to the relation

"for all $i \in I$, $f_i \circ f : |B| \to |A_i|$ is a $\mathcal{C}$-morphism".

Obviously, the source $(f_i : A \to A_i)_{i \in I}$ is initial if, and only if for every object A' with the same structure as A, i.e. for which there exists a $\mathcal{C}$-isomorphism $\varphi : A' \to A$ with $|\varphi| = \mathrm{id}_{|A|}$, the source $(f_i \circ \varphi : A' \to A_i)_{i \in I}$ is initial.

Definition 6 *If the source $(f_i : A \to A_i)_{i \in I}$ is initial, then the structure of A is called initial with respect to the family $(f_i : |A| \to |A_i|)_{i \in I}$.*

Theorem 7 *If the structure $\mathfrak{S}$ of A is initial for the family $(f_i : |A| \to |A_i|)_{i \in I}$, $\mathfrak{S}$ is the coarsest of all structures $\mathfrak{S}'$ of objects A' of $\mathcal{C}$ such that all $\mathcal{X}$-morphisms $f_i : |A'| \to |A_i|$ $(i \in I)$ are $\mathcal{C}$-morphisms, and consequently is unique.*

Proof. Assume that the structure $\mathfrak{S}$ of A is initial for the family $(f_i : |A| \to |A_i|)_{i \in I}$. Then all $f_i : |A| \to |A_i|$ $(i \in I)$ are $\mathcal{C}$-morphisms. Let A' be such all $f_i : |A'| \to |A_i|$ are $\mathcal{C}$-morphisms. Since f_i is an $\mathcal{X}$-morphism, $|A'| = |A|$ and $f = \mathrm{id}_{|A|}$ is an $\mathcal{X}$-morphism and for each $i \in I$, $f_i \circ f : |A'| \to |A_i| = f_i$ is a $\mathcal{C}$-morphism. Thus $f : |A'| \to |A|$ is a $\mathcal{C}$-morphism and $\mathfrak{S}$ is coarser than the structure of A'. This determines $\mathfrak{S}$ in a unique way by Lemma 4. ■

As shown by ([2], p. IV.30, exerc. 30), it may happen that a structure $\mathfrak{S}$ in a concrete category $(\mathcal{C}, |.|, \mathcal{X})$ be the coarsest of all structures $\mathfrak{S}'$ of objects A' of $\mathcal{C}$ such all $\mathcal{X}$-morphisms $f_i : |A'| \to |A_i|$ are $\mathcal{C}$-morphisms, and that this structure be not initial with respect to the family $(f_i : |A| \to |A_i|)_{i \in I}$.

The opposite of the concrete category $(\mathcal{C}, |.|, \mathcal{X})$ is $(\mathcal{C}^{\mathrm{op}}, |.|^{\mathrm{op}}, \mathcal{X}^{\mathrm{op}})$ where for each pair of objects A, B of $\mathcal{C}$, $|A \to B|^{\mathrm{op}} = |A \leftarrow B|$. The structure of A is called final with respect to the family $(f_i : |A| \longleftarrow |A_i|)_{i \in I}$ in $(\mathcal{C}, |.|, \mathcal{X})$ if it is initial with respect to this family in $(\mathcal{C}, |.|, \mathcal{X})$.

3 Applications

Examples of initial structures ([2], § 2.4): inverse image of a structure, induced structure, concrete product (see Remark 8).

Examples of final structures ([2], § 2.6): direct image of a structure, quotient structure, concrete coproduct (see Remark 8).

Remark 8 *Let $\mathcal{P} = (\mathrm{pr}_i : A \to A_i)_{i \in I}$ be a product in $\mathcal{C}$. This product is called concrete in $(\mathcal{C}, |.|, \mathcal{X})$ if $|\mathcal{P}| = (|\mathrm{pr}_i| : |A| \to |A_i|)_{i \in I}$ is a product in $\mathcal{X}$ ([1], §10.52). All products considered in ([2], § 2.4) are concrete but there exist concrete categories with non-concrete products ([1], §10.55(2)). Dually, there exist concrete categories with non-concrete coproducts ([1], §10.67).*

4 References

References

[1] J. Adámek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories*, John Wiley, 1990 (new edition: J. Adámek, Horst Herrlich, and George E. Strecker, 2004).

[2] N. Bourbaki, *Éléments de mathématique. Livre I : Théorie des ensembles*, Hermann, 1970.