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Quasi- poles of linear time-varying systems in an intrinsic algebraic approach

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Abstract

In a previous piece of work it has been shown that exponential stability of a linear time-varying (LTV) system can be evaluated using new definitions of the *poles* of such a system. The latter are given by a *fundamental set of roots* of the skew polynomial $P(\partial)$ which defines the autonomous part of the system. Such a set may not exist over the initial field $\mathbf{K}$ of definition of the coefficients of the system, but can exist over a suitable field extension $\tilde{\mathbf{K}} \supset \mathbf{K}$. It is shown here that conditions for stability can also be obtained using linear factors of the polynomial $P(\partial)$ over another field extension $\check{\mathbf{K}}$ which may be smaller: $\tilde{\mathbf{K}} \supset \check{\mathbf{K}} \supset \mathbf{K}$. The roots of these factors are called the *quasi-poles* of the system. The necessary condition for system stability, expressed in function of these quasi-poles, is more restrictive than the one involving a fundamental set of roots.

Keywords: poles, zeros, exponential stability, linear time-varying systems, factorisation of a skew-polynomial, field extensions

1. Introduction

The poles of a system are important since they provide a direct way for the analysis of the stability of the system. Several kinds of poles have been recently introduced for linear time-varying (LTV) systems (Linear Time Invariant (LTI) systems are considered as a special case of LTV systems in this paper). In [12], Lyapunov transformations were used to obtain an upper triangular form of the state matrix, from which the notion of *pole set of the state matrix* was defined. Other approaches consider the system in a more intrinsic way, as a finitely presented module M over a ring of differential operators or, equivalently, a ring of skew polynomials in indeterminate $\partial = d/dt$. This algebraic framework was initiated by Malgrange [9] and next exploited in systems theory by several authors (see [4], [11], [14], [13], [2] and related references). The same algebraic framework has been exploited in [10] to define poles and zeros along with their multiplicities for LTV systems. In the latter contribution, the poles were defined using what was called a *fundamental set of right roots of the skew polynomial* $P(\partial)$ which defines the autonomous part of the system M. More precisely, the latter is given by a scalar differential equation of the form

$$P(\partial)y = 0,\ t \geq t_0,\ \ P(\partial) = \sum_{i=0}^{n} a_i \partial^i,\ n \in \mathbb{N},\ a_i \in \mathbf{K} \tag{1}$$

where $\mathbf{K}$ is a field which contains not only constants, i.e., $\exists a \in \mathbf{K}$ such that $\partial a \neq 0$. It was shown that such a set of roots leads to a factorization

$$P(\partial) = (\partial - a_n)...(\partial - a_1). \tag{2}$$

where the a_i's are pairwise nonconjugated. Factorization (2) is obtained over a well-chosen field extension of the initial field $\mathbf{K}$ of definition of the coefficients of M: $a_i \in \tilde{\mathbf{K}} \supset \mathbf{K}$, $i = 1, ..., n$ [15].

In the work presented here it is shown that, under certain conditions, exponential stability can also be evaluated from a factorization

$$P(\partial) = (\partial - a_m)^{d_m}...(\partial - a_1)^{d_1}, m \le n \tag{3}$$

where the a_i's are pairwise nonconjugated. Notice that (3) is more general than (2) and therefore can be obtained over a field $\check{\mathbf{K}}$ such that $\tilde{\mathbf{K}} \supset \check{\mathbf{K}} \supset \mathbf{K}$. In what follows, stability of the autonomous system is shown to be related to the zeros a_i of P of a factorization of type (3). Thus, in an control theory setting, the a_i's are called *quasi-poles* in the sequel.

The paper is organized as follows: Section 2 introduces the mathematical background, especially facts about field extensions, skew polynomials and differential operators which are well-known within the communities of ordinary differential equations and symbolic computation (see, e.g., [3], [15], [5] and related references). In Section 3, the definitions given in [10] along with some key results on the poles and zeros of LTV systems, are recalled. The link between the multiple factors of (3) and the solutions of differential equation (1) is shown in Sections 4 and 5 while Section 6 is devoted to concluding remarks.

2. Background notions

2.1. LTV systems and behaviors

In the algebraic framework mentioned in the Introduction, a linear system is a finitely presented left module M over the ring $\mathbf{R} = \mathbf{K}[\partial]$ of ordinary differential operators in $\partial = d/dt$ with coefficients in a differential field $\mathbf{K}$ (i.e., a commutative field equipped with one derivation). If $\mathbf{K}$ does not exclusively contain constants (i.e., elements whose derivative is zero), M is an LTV system. A vector of inputs of M is a set of elements $u = (u_i)_{i=1,...,m}$ such that $[u]_\mathbf{R}$, the $\mathbf{R}$-module generated by the entries of u, is free of rank m and $M/[u]_\mathbf{R}$ is *torsion*. Notice that $[u]_\mathbf{R}$ is a maximal free $\mathbf{R}$-submodule of M. If an output $y = (y_i)_{i=1,...,p}$, $y_i \in M$[1] is also chosen, the triple (M, u, y) is a *control system*.

Let M be defined by $Rw = 0$ where R is a matrix with entries in $\mathbf{R}$. Considering the solutions w.r.t. time, the entries of w, which define the variables of the system, must belong to a well-defined space of functions (or distributions, hyperfunctions, etc.) W. The object $ker_W(R\bullet)$, where $R\bullet$ denotes the left-multiplication by R, is called the $W-behavior$ associated with M. It is denoted by $\mathfrak{B}_W(M)$ and is a left $\mathbf{R}$-module. Given W, $\mathfrak{B}_W(M)$ is deduced from M using the functor $\mathfrak{B}_W(\bullet) = Hom_\mathbf{R}(\bullet, W)$ [9]. Conversely, M is uniquely determined by $\mathfrak{B}_W(M)$ if W is a *cogenerator* left $\mathbf{R}$-module [11].

Let f be a real-valued analytic function defined in an interval of $\mathbb{R}$ of the form $(A, +\infty)$ for some real A. If $lim_{t\to+\infty} f(t) = +\infty$, $\dot{f}(t) > 0$, $t > B$ for some real $B \ge A$ and $lim_{t\to+\infty} \frac{\dot{f}(t)}{f(t)} = 0$, then f is called an *Ore function*. A complex (resp. real) Ore field is a field $\mathbb{C}(f)$ (resp. $\mathbb{R}(f)$) where f is an Ore function (i.e., a field of rational functions in f with coefficients in $\mathbb{C}$ (resp. $\mathbb{R}$)). As shown in [2], every function g belonging to an Ore field is such that:

- $lim_{t\to+\infty} Re(g(t))$ and $lim_{t\to+\infty} Im(g(t))$ exist in $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$
- is infra-exponential (i.e., $lim_{t\to+\infty} e^{-\alpha t} g(t) = 0$) for every $\alpha > 0$)
- $lim_{t\to+\infty} \frac{\dot{g}(t)}{g(t)} = 0$
- if this Ore field is real, $g(t)$ has a constant sign as $t \to +\infty$.

Let $\mathcal{O}_\infty$ be the space of germs of analytic functions defined in an interval $(A, +\infty)$ as above. Assume that $\mathbf{K}$ is an Ore field. Then, the left $\mathbf{K}[\partial]$-module $\mathcal{O}_\infty$ is an injective cogenerator [2].

[1]The vector y here should not be mixed up with the scalar variable y in (1)

2.2. Autonomous systems

The so-called *autonomous part* Σ of the system M is obtained by factoring out the input variables. From the algebraic point of view this corresponds to the *torsion module* $T \cong M/[u]_{\mathbf{R}}$. As $\mathbf{R}$ is a simple ring[2], T is a cyclic left $\mathbf{R}$-module (see, e.g., [8]) and can be characterized by a state representation $\dot{x} = Ax$ (where $\dot{x} = \frac{dx}{dt}$), $A \in \mathbf{K}^{n\times n}$ or, equivalently, by (1) with $P(\partial) = \partial^n + \sum_{i=0}^{n-1} p_i\partial^i$ and y a generator of T. In the latter case, $T \cong \mathbf{R}/\mathbf{R}P$. Notice that equation (1) is not the unique differential equation which defines the autonomous system. Indeed, if $\overline{T} \cong T$ is an $\mathbf{R}$-module defined by the equation $\overline{P}(\partial)\overline{y} = 0$, then $P(\partial)$ and $\overline{P}(\partial)$ are said to be *similar* (written $\overline{P}(\partial) \sim P(\partial)$) (see, e.g., [3]) and they thus define the *same* autonomous system. Then $\mathbf{R}/\mathbf{R}P(\partial) \cong \mathbf{R}/\mathbf{R}\overline{P}(\partial)$ (as modules) and $Hom_{\mathbf{R}}(\mathbf{R}/\mathbf{R}P(\partial), \mathcal{O}_\infty) \cong Hom_{\mathbf{R}}(\mathbf{R}/\mathbf{R}\overline{P}(\partial), \mathcal{O}_\infty)$ (as $\mathbb{C}-$vector spaces). Conversely, as $\mathcal{O}_\infty$ is an injective cogenerator, from the latter isomorphism it follows that $\mathbf{R}/\mathbf{R}P(\partial) \simeq \mathbf{R}/\mathbf{R}\overline{P}(\partial)$. Stability can thus be analysed from the polynomial $P(\partial)$ in an equation (1) which defines that autonomous system T.

The autonomous LTV system given by the torsion module T is said to be

- *exponentially stable* if any $\mathcal{O}_\infty$-solution approaches zero exponentially for $t \to +\infty$, i.e., $\forall y \in Hom_{\mathbf{R}}(T, \mathcal{O}_\infty)$ $y : t \mapsto y(t)$ of (1), $\exists C > 0$ and $\exists \tau > 0$ such that $\mid y(t) \mid \leq Ce^{-\tau t}$ for large enough $t > 0$.
- *exponentially unstable* if there exists a $\mathcal{O}_\infty$-solution $y :\ t \mapsto y(t)$ of (1) which is exponentially unbounded.

2.3. Noncommutative algebra

When dealing with LTV systems, polynomial $P(\partial)$ in (1) is *skew*, i.e., belongs to the *noncommutative* ring $\mathbf{R} = \mathbf{K}[\partial]$ equipped with the commutation rule

$$\partial a = a\partial + \dot{a} \tag{4}$$

which is the Leibniz rule of derivation of a product [3]. Let $P(\partial) = \sum_{i=1}^{n} a_i\partial^i$. If $\partial - \alpha$ is a right factor of $P(\partial)$ we say that α is a *(right) root* of P. In this case, there exists a polynomial $Q(\partial)$ such that $P(\partial) = Q(\partial)(\partial - \alpha)$ and choosing y to be a nonzero solution of $\partial y = \alpha\partial$ (e.g., $y(t) = e^{\int_{t_0}^{t} \alpha(\tau)d\tau}$ in this expression is well-defined), then

$$\begin{aligned} &P(\partial)y = \textstyle\sum_{i=0}^{n} a_i\partial^i y = a_0y_0 + a_1\alpha y + a_2\partial\alpha y + a_3\partial(\partial\alpha y) + ... \\ &= a_0y_0 + a_1\alpha y + a_2(\alpha^2 + \dot{\alpha})y + a_3\partial(\alpha^2 + \dot{\alpha})y + ... \\ &= \textstyle\sum_{i=0}^{n}(a_iN_i(\alpha))y, \end{aligned} \tag{5}$$

where

$$N_0(\alpha) = 1 \; and \; N_{i+1}(\alpha) = \alpha N_i(\alpha) + \dot{N}_i(\alpha) \; for \; i \geq 0 \tag{6}$$

and thus $\sum_{i=0}^{n} a_iN_i(\alpha) = 0$ since $P(\partial)y = 0$. The last expression in (5) gives the *evaluation of P at $\alpha \in \boldsymbol{K}$*, denoted by $P(\alpha)$:

$$P(\alpha) = \sum_{i=0}^{n} a_iN_i(\alpha) \tag{7}$$

where N_i is defined inductively by (6). $P(\alpha) = 0$ if, and only if (iff), $\partial - \alpha$ is a right factor of $P(\partial)$. Notice that it is difficult in practice to compute the roots of P from the equation $P(\alpha) = 0$ since expression (7) of the evaluation of P at α leads to highly nonlinear ordinary differential equations in α.

If there exist polynomials $P'(\partial)$, $P''(\partial)$ and an element $a \in \mathbf{K}$ such that $P(\partial) = P'(\partial)(\partial - a)P''(\partial)$, then a is called a *zero* of P. If P has a factorization (2) into n linear factors $\partial - a_i$ (not necessarily distinct), then $\{a_1, ..., a_n\}$ is called a *full set of zeros* of $P(\partial)$. For any $0 \neq c \in \mathbf{K}$, the *conjugate of α by c*, denoted by α^c, is $\alpha^c = c\alpha c^{-1} + \dot{c}c^{-1} = \alpha + \dot{c}c^{-1}$ [3] (since $\mathbf{K}$ is a field, thus $\dot{c}$ and c commute) and can be interpreted as follows: the variable y satisfies $\dot{y} = \alpha y$ iff $z \triangleq cy$ satisfies $\dot{z} = \alpha^c z$, i.e. the multiplication by c is an isomorphism $\mathbf{R}/\mathbf{K}\,(\partial - \alpha) \tilde{\rightarrow} \mathbf{R}/\mathbf{K}\,(\partial - \alpha^c)$. $\Delta_{\mathbf{K}}(\alpha) = \{\alpha^c,\ c \in \mathbf{K}, c \neq 0\}$ is called the *conjugacy class* of α over $\mathbf{K}$. The least

[2]i.e., a ring which has no proper nonzero ideal

common left multiple $[P_i, 1 \leq i \leq n]_l$ of skew polynomials P_i $(1 \leq i \leq n)$ exists. Let $\Delta = \{\gamma_1, ..., \gamma_n\} \subseteq \mathbf{K}$. The *minimal polynomial of* Δ, denoted by P_Δ is $P(\partial)_\Delta = [\partial - \gamma_i, 1 \leq i \leq n]_l$. Δ is called P*-independent* if the degree of P_Δ is n. A skew polynomial P of degree n has an infinite number of *right roots* which lie in at most n conjugacy classes (see, e.g., [3] and [5]). All polynomials used in the sequel are skew, so all roots involved are right roots. To alleviate the presentation, both adjectives will be skipped from now on.

3. Poles and stability [10]

3.1. Fundamental sets of roots

As shown in the section above, the definition of right roots provides a direct link between the roots of polynomial $P(\partial)$ and the solutions of differential equation (1). In order to obtain *the whole class* of solutions of (1), a *fundamental set of roots* of $P(\partial)$ must be considered.

Definition 1: If $P(\partial)$ is a polynomial of degree n, a set of n P-independent roots of $P(\partial)$[3] is called a *fundamental set of roots* of $P(\partial)$. A polynomial which has a fundamental set of roots over the field **K** of definition of its coefficients is called a *Wedderburn* or W-polynomial (over **K**). $\diamond$

Example 1: $P(\partial) = (\partial - a)^2$, $a \in \mathbb{R}$. Obviously a is a root, but the two roots $\{a, a\}$ are not P-independent. From the double root a one can compute a fundamental set of roots over, e.g., $\mathbb{C}(t)$, following the procedure introduced in [10] : this set is $\{a, a + t^{-1}\}$. $\diamond$

The elements γ_i of a fundamental set of roots are in direct relation with the elements y_i of a *fundamental set of solutions* of the differential equation (1). These relations consists in the elementary equations[4]

$$\dot{y}_i y_i^{-1} = \gamma_i. \tag{8}$$

For Example 1, obviously, $y_1 = e^{at}$, $y_2 = te^{at}$, is a *fundamental set of solutions* of (1).

To further explain the link between a fundamental set of roots $\{\gamma_1, ..., \gamma_n\}$ of $P(\partial)$ and a fundamental set of solutions $\{y_1, ..., y_n\}$ of equation (1) let

$$W(y_i)_{i=1}^n = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{dy_1}{dt} & \frac{dy_2}{dt} & \cdots & \frac{dy_n}{dt} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{d^{n-1}y_1}{dt} & \frac{d^{n-1}y_2}{dt} & \cdots & \frac{d^{n-1}y_n}{dt} \end{bmatrix}, \quad V(\gamma_i)_{i=1}^n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \partial(\gamma_1) & \partial(\gamma_2) & \cdots & \partial(\gamma_n) \\ \partial^2(\gamma_1) & \partial^2(\gamma_2) & \cdots & \partial^2(\gamma_n) \\ \cdots & \cdots & \cdots & \cdots \\ \partial^{(n-1)}(\gamma_1) & \partial^{(n-1)}(\gamma_2) & \cdots & \partial^{(n-1)}(\gamma_n) \end{bmatrix} \tag{9}$$

be the Wronskian and, respectively, the Vandermonde matrix associated with the two sets mentioned above. The following result is a particularization to the conjugacy class of $0 \in \mathbf{K}$ of a more general property established for W-polynomials in [6] and related references.

Proposition 1: For a set $\{y_1, \ldots, y_n\}$ in any differential field **K**, let $\gamma_i = \dot{y}_i y_i^{-1}, \ i = 1, ..., n$ and consider the Vandermonde matrix of $\gamma_1, \ldots, \gamma_n$ defined by V in (9) where $\partial^i(.)$ is the evaluation of the polynomial ∂^i as introduced in Section 2.3. Then

1. $W(y_i)_{i=1}^n = V(\gamma_i)_{i=1}^n diag\{y_1, \ldots, y_n\}$.
2. The following conditions are equivalent:
   (i) $W(y_i)_{i=1}^n$ (or, equivalently, $V(\gamma_i)_{i=1}^n$) is invertible
   (ii) The set $\{\gamma_1, \ldots, \gamma_n\}$ is P-independent.

[3] according to the definition and notation in [5]

[4] γ_i is thus the logarithmic derivative of y_i.

$\diamond$

Definition 2: Consider an autonomous LTV system Σ given by a torsion **R**-module $T \cong \mathbf{R}/\mathbf{R}P(\partial)$ where $\mathbf{R} = \mathbf{K}[\partial]$ and let $\gamma_i \in \widetilde{\mathbf{K}} \supseteq \mathbf{K}$, $i = 1, ..., n$, be such that $\{\gamma_1, ..., \gamma_n\}$ is a fundamental set of roots of $P(\partial)$. If $\widetilde{\mathbf{K}}$ is an Ore field, the γ_i's are called the *poles* of Σ; $\{\gamma_1, ..., \gamma_n\}$ is then called a *fundamental set of poles* of Σ. $\diamond$

Theorem 1 [[10]]: Consider an autonomous system $\sum$ which is defined by the **R**-torsion module $T \cong \mathbf{R}/\mathbf{R}P(\partial)$. Let $\{\gamma_1, ..., \gamma_n\}$ be a fundamental set of poles of $\sum$ over an Ore field $\widetilde{\mathbf{K}}$ (assuming that such a set exists). Then, $\sum$ is *exponentially stable* iff, for all $i \in \{1, ..., n\}$,

$$\lim_{t \to +\infty} Re\{\gamma_i(t)\} < 0. \tag{10}$$

$\sum$ is *exponentially unstable* iff at least one of the γ_i's satisfies the condition

$$\lim_{t \to +\infty} Re\{\gamma_i(t)\} > 0. \diamond \tag{11}$$

Notice that, since $\tilde{\mathbf{K}}$ is an Ore field, the limits (22) and (23) exist in $\bar{\mathbb{R}}$ and depend only on the conjugacy class of γ_i. Notice also that a fundamental set of roots of P does not always exist over **K**, the initial field of definition of the system; in that case, a field extension $\widetilde{\mathbf{K}}$ must be constructed [10]. This is the case of Example 1: $\mathbf{K} = \mathbb{R}$ but $\widetilde{\mathbf{K}} = \mathbb{R}(t) \supsetneq \mathbb{R}$.

Example 2: Let $P(\partial) = \partial^2 + (t^{-1} - 2a)\partial + a^2 - t^{-2} - at^{-1}$, $a \in \mathbb{R}$. We can assume that $\mathbf{K} = \mathbb{R}(t)$. A fundamental set of solutions of (1) with $P(\partial)$ as above is $y_1 = te^{at}$, $y_2 = t^{-1}e^{at}$ and a fundamental set of roots of $P(\partial)$ is $\gamma_1 = a + t^{-1}$, $\gamma_2 = a - t^{-1}$. Since $\gamma_1, \gamma_2 \in \mathbb{R}(t)$, $P(\partial)$ is a W-polynomial over the initial field of definition of the LTV system, i.e., $\widetilde{\mathbf{K}} = \mathbf{K}$, and no field extension is needed in this case.

4. Multiple factors

In the preceding section, a relation between a set of n P-independent (right) roots of polynomial $P(\partial)$ and the solutions of the differential equation (1) has been recalled using the elementary equations (8). This obviously allows one to conclude on the asymptotic behavior of the solutions y_i of (1) by investigating the roots γ_i as stated in Theorem 1. However, this analysis can also be done in many cases with a factorization (2) of $P(\partial)$ for which there exist indices $i, j \in \{1, ..., n\}$ such that a_i and a_j are conjugated. In this case, the polynomial $P(\partial)$ in (2) is not necessarily a W-polynomial over the field **K** of definition of the elements a_i.

Theorem 2: Consider the homogeneous differential equation (1).

1. Suppose $\{a_1, ..., a_n\}$ is a full set of zeros of $P(\partial)$. If

$$limsup_{t \to +\infty} Re\{a_i(t)\} < 0, \; i = 1, ..., n \tag{12}$$

then any solution of (1) approaches zero exponentially as $t \to +\infty$.

2. If $P(\partial)$ admits a factorization (2) for which all a_i's belong to a *real* Ore field, then condition (12) is also *necessary* for all the solutions of (1) to approach zero exponentially zero as $t \to +\infty$. In the latter situation, condition

$$lim_{t \to +\infty} Re\{a_i(t)\} > 0 \; for \; some \; i \in \{1, ..., n\} \tag{13}$$

is necessary and sufficient for the existence of an exponentially unbounded solution of (1).

3. Assume that $P(\partial) \in \mathbf{K}[\partial]$ where $\mathbf{K}$ is an Ore field and $P(\partial)$ admits a factorisation (3) where the a_i's are pairwise non conjugated. Then,

$$T = \oplus_{1 \leq i \leq m} T_i, \quad T_i \cong \frac{\mathbf{R}}{\mathbf{R}(\partial - a_i)^{d_i}}. \tag{14}$$

where $T = \mathbf{R}/\mathbf{R}P(\partial)$. Moreover, we know that $\lim_{t\to+\infty} Re\{a_i(t)\}$ exists in $\bar{\mathbb{R}}$. If there exists $i \in \{1, ..., n\}$ such that $\lim_{t\to+\infty} Re\{a_i(t)\} = 0$ (resp., $\lim_{t\to+\infty} Re\{a_i(t)\} > 0$), then there exists a solution y of (1) which is hypo-exponential and such that $|y(t)| \nrightarrow 0$ as $t \to +\infty$ (resp., $|y(t)| \to +\infty$ exponentially as $t \to +\infty$).

Proof:

1. Equation (1) with $P(\partial)$ given by (2) can equivalently be written in the form

$$\begin{cases} (\partial - a_1)y = z_1 \\ (\partial - a_2)z_1 = z_2 \\ \vdots \\ (\partial - a_{n-1})z_{n-2} = z_{n-1} \\ (\partial - a_n)z_{n-1} = 0. \end{cases}, t \geq t_0 \tag{15}$$

Set $z_0 = y$ and suppose that (12) holds. The dynamics of z_{n-1} depends only on the last equation of (15) which is of elementary type (8):

$$z_{n-1}(t) = \lambda_n e^{\int_{t_0}^{t} a_n(\tau)d\tau} \tag{16}$$

where λ_n is any constant. As a_n satisfies (12), $Re\{a_n(t)\} \leq -\beta_n$ for some $\beta_n > 0$ and for t large enough, say, $t \geq t_1$. It follows that

$$|e^{\int_{t_1}^{t} a_n(\tau)d\tau}| \leq e^{-\beta_n(t-t_1)}, \ \forall t \geq t_1 \tag{17}$$

and, as a consequence,

$$|z_{n-1}(t)| \leq \alpha_n e^{-\beta_n(t-t_1)}, \forall t \geq t_1 \tag{18}$$

where $\alpha_n = |\lambda_n| e^{\int_{t_0}^{t_1} a_n(\tau)d\tau}$. Thus $z_{n-1}(t)$ approaches zero exponentially as $t \to +\infty$. The rest of the variables z_i, $i = n-2, ..., 0$ have the same property. Indeed, using Lemma 1 in Appendix A and the fact that a_{n-1} satisfies (12), z_{n-2}, the solution of the last but one equation of (15), approaches zero exponentially as $t \to +\infty$. The conclusion follows by induction towards the first equation of (15).

2. Let now the a_i's belong to a real Ore field. Then $lim_{t\to+\infty} a_i(t)$ exists, so let $lim_{t\to+\infty} a_i(t) = \overline{a}_i$, $i = 1, ..., n$. Suppose that all solutions of (1) approach zero exponentially when $t \to +\infty$. A particular solution of (1) or, equivalently, of (15), is $y(t) = y_1(t) = e^{\int_{t_0}^{t} a_1(\tau)d\tau}$, $z_1(t) = z_2(t) = ... = z_{n-1}(t) = 0$. More specifically, y_1 corresponds to the solution of the homogeneous part of the first equation of (15). Moreover, $lim_{t\to+\infty} y_1(t) = e^{\int_{t_0}^{+\infty} a_1(\tau)d\tau}$. As $y_1(t)$ approaches zero exponentially as $t \to +\infty$, there exist $\alpha > 0$ and $t_1 > t_0$ such that $\left| e^{\int_{t_0}^{t}(a_1(\tau)+\alpha)d\tau} \right| \leq 1$ whenever $t \geq t_1$. Therefore, for $t \geq t_1$, $\int_{t_0}^{t} (a_1(\tau) + \alpha) d\tau \leq 0$. We know that $lim_{t\to+\infty} a_1(t) = \bar{a}_1$, therefore there exists $t_2 \geq t_1$ such that $|a_1(t) - \bar{a}_1| \leq \frac{\alpha}{2}$ whenever $t \geq t_2$. For these values of t, $\bar{a}_1 \leq -\frac{\alpha}{2} + \frac{1}{t-t_2} \int_{t_0}^{t_2} (a_1(\tau) + \alpha) d\tau$. Taking $t \to +\infty$ yields $\bar{a}_1 \leq -\frac{\alpha}{2} < 0$. Moreover, from (16) it follows that the sign of z_{n-1} is constant as $t \to +\infty$. Next, the solutions of (15) are

$$z_{k-1}(t) = ce^{\int_{t_0}^{t} a_k(\tau)d\tau} + e^{\int_{t_0}^{t} a_k(\tau)d\tau} \int_{t_0}^{t} z_k(\tau) e^{-\int_{t_0}^{\tau} a_k(\alpha)d\alpha} d\tau \tag{19}$$

where c is any constant and $k = 2, ..., n$. It follows by induction that the signs of z_{n-2}, ..., z_1 are also constant as $t \to +\infty$. Using now Lemma 2 in Appendix A for the first equation in (15) one concludes that $z_1(t)$

decreases exponentially to 0 when $t \to +\infty$. Another particular solution of (15) is $y(t)$, $z_1(t) = e^{\int a_2(t)dt}$, $z_2(t) = z_3(t) = ... = z_{n-1}(t) = 0$. It follows, as before, that $\bar{a}_2 < 0$. Again, using Lemma 2 in Appendix A for the second equation in (15) one concludes that $z_2(t)$ decreases exponentially to 0 when $t \to +\infty$. The conclusion follows using the same rationale down to the last equation.

For the last part of this point, by (16), $\chi(z_{n-1}) = \bar{a}_n$, where $\chi(.)$ denotes the Lyapunov exponent defined in Appendix A ($\chi(f) = limsup_{t\to+\infty} \frac{ln|f(t)|}{t}$). Also, by (19), $z_{n-2} = cz'_{n-2} + z''_{n-2}$ where

$$z'_{n-2} = e^{\int_{t_0}^t a_{n-1}(\tau)d\tau},\ z''_{n-2} = e^{\int_{t_0}^t a_{n-1}(\tau)d\tau} \int_{t_0}^t z_{n-1}(\tau) e^{-\int_{t_0}^\tau a_{n-1}(\alpha)d\alpha} d\tau. \tag{20}$$

From Lemma 3 (a) in Appendix A, $z'_{n-2} \in Exp(\bar{a}_{n-1})$. From parts (a) and (b) of the same lemma, $z_{n-1}(t)e^{-\int_{t_0}^t a_{n-1}(\tau)d\tau} \in Exp(\bar{a}_n - \bar{a}_{n-1})$. From part (c) of the same result, $\int z_{n-1}(t)e^{-\int a_{n-1}(t)dt}dt \in Exp(\bar{a}_n - \bar{a}_{n-1})$; again, from part (b), $z''_{n-2} \in Exp(\bar{a}_n)$ and, choosing $c > 0$, $z_{n-2} \in Exp(sup(\bar{a}_{n-1}, \bar{a}_{n-2}))$. The result follows by induction.

3. As a_1 in (3) is not conjugated with a_j for $j \in \{2, ..., m\}$, a_1 is not a left root of $P_{m-1}(\partial) = (\partial - a_m)^{d_m}...(\partial - a_2)^{d_2}$ and, therefore, $(\partial - a_1)^{d_1}$ and $P_{m-1}(\partial)$ are left-coprime. By the chinese remainder theorem [7], it follows that

$$\frac{\mathbf{R}}{\mathbf{R}P(\partial)} \cong \frac{\mathbf{R}}{\mathbf{R}P_{m-1}(\partial)} \oplus \frac{\mathbf{R}}{\mathbf{R}(\partial - a_1)^{d_1}} \tag{21}$$

and (14) follows by induction. The second part of this point is a consequence of Lemma 4.
$\diamond$

Example 3: Consider (1) with $P(\partial) = (\partial - a_2)(\partial - a_1)$ where $a_1 = -t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} + \alpha$, $a_2 = -t^{-\frac{3}{2}} + \frac{3}{2}t^{-1} + t + \alpha$, $\alpha \in \mathbb{R}$. Thus, $\check{\mathbf{K}} = \mathbb{R}(t)$. From Theorem 2 one can conclude that the solutions of (1) approach zero exponentially iff $\alpha < 0$. If $\alpha > 0$, one of them diverges exponentially to $+\infty$. Let now compute from the full set of zeros of P $\{a_1, a_2\}$ a fundamental set of roots $\{\gamma_1, \gamma_2\}$. Obviously, $\gamma_1 = a_1$. Following the procedure given in [10], one can find $\gamma_2 = -t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} + \alpha + \frac{1}{e^{\frac{t^2}{2}} \int_{t_0}^t e^{\frac{-t^2}{2}} dt}$ from which it follows that $\check{\mathbf{K}} \subsetneq \widetilde{\mathbf{K}}$.

5. Quasi-poles and stability

Definition 3: Consider an autonomous LTV system Σ given by a torsion $\mathbf{R}$-module $T \cong \mathbf{R}/\mathbf{R}P(\partial)$ where $\mathbf{R} = \mathbf{K}[\partial]$ and let $a_i \in \check{\mathbf{K}} \supseteq \mathbf{K}$, $i = 1, ..., n$, be such that $\{a_1, ..., a_n\}$ is a full set of zeros of $P(\partial)$. Let $\pi_1, ..., \pi_p$ ($1 \le p \le n$) be the conjugacy classes of the a_i's. The *multiplicity* of π_j is the number ν_j of zeros a_i which belong to π_j. Therefore, $\sum_{i=1}^p \nu_i = n$. If $\check{\mathbf{K}}$ is an Ore field over which there exists a direct sum decomposition (14), the π_i's are called the *quasi-poles* of Σ; $\{a_1, ..., a_n\}$ is then a *full set of representants of quasi-poles* of Σ. $\diamond$

The following result is a consequence of Theorem 2 exploited in the module framework given in Section 2.2, taking into account the asymptotic behavior of the elements of conjugacy classes over Ore fields:

Theorem 3 (main result): Consider an autonomous system $\sum$ which is defined by the $\mathbf{R}$-torsion module $T \cong \mathbf{R}/\mathbf{R}P(\partial)$.

- Let $\{a_1, ..., a_n\}$ be a full set of representants of quasi-poles of $\sum$ over an Ore field (assuming that such a set exists). Then, $\sum$ is *exponentially stable* iff, for all $i \in \{1, ..., n\}$,

$$\lim_{t\to+\infty} Re\{a_i(t)\} < 0. \tag{22}$$

$\sum$ is *exponentially unstable* iff at least one of the a_i's satisfies the condition

$$\lim_{t\to+\infty} Re\{a_i(t)\} > 0. \tag{23}$$

- If $P(\partial)$ has a full set of zeros $\{a_1, ..., a_n\}$ in an Ore field, then $\sum$ is *exponentially stable* if (22) holds for all $i \in \{1, ..., n\}$. If the a_i's belong to a *real* Ore field, condition (22) is also necessary for exponential stability; furthermore, condition (23) is also necessary for exponential instability. $\diamond$

Remark: Notice that conditions (22) and (23) are expressed using a full set of representants of quasi-poles obtained for a given representation of the system $\sum$ (i.e., a given equation (1)), but, according to the module framework recalled in Section 2, they are intrinsic in the sense that they depend only on the quasi-poles themselves.

6. Conclusion

Conditions for exponential stability have been given in terms of system quasi-poles (Theorem 3). Both poles and quasi-poles of a given LTV system are calculated from factorizations of the polynomials which define the autonomous part of the system. The latter factorizations may not exist over the initial field of definition of the coefficients of the system and, in this case, field extensions are needed. In general, the quasi-poles can be computed over a smaller field extension than the poles. If quasi-poles cannot be defined, sufficient conditions for exponential stability are given using a full set of zeros (Theorem 3). If the latter are real-valued, these sufficient conditions are also necessary.

The definition and analysis of quasi-poles of the system can be extended to the zeros and hidden modes of the system in a similar way. Indeed, these entities have been defined as torsion modules for LTI systems in [1]. In [10] it has been shown that these definitions can be extended to the LTV case. This framework holds also for the quasi-entities (quasi-poles and -zeros) introduced here.

Further extensions of this work will concern algorithms to provide factorizations of type (3). This is a difficult task and, despite many important works in the direction of factoring ordinary differential operators (like [15] and related references), it is not always possible to obtain such factorizations.

Appendix A. Four lemmas

The proofs of these lemmas are given in [2]; only some hints are provided here.

Lemma 1: Consider the equation

$$(\partial - a)y = z,\ a, z \in \mathcal{O}_\infty. \tag{A.1}$$

If $\limsup_{t\to+\infty} Re\,(a\,(t)) < 0$ and $z\,(t) \to 0$ exponentially as $t \to +\infty$, then so does y. $\diamond$

Sketch of the proof : The solutions of $(A.1)$ are

$$y(t) = cy_1(t) + y_2(t),\ y_1(t) = e^{\int adt},\ y_2(t) = y_1(t)\int ze^{-\int adt}dt \tag{A.2}$$

where c is any constant. As $\limsup_{t\to+\infty} Re\,(a\,(t)) < 0$, from (A.2) follows that $y_1(t)$ approaches zero exponentially when $t \to +\infty$. As $z\,(t) \to 0$ exponentially as $t \to +\infty$ there exists $\beta > 0$ be such that $|z| = O\left(e^{-\beta t}\right)$ as $t \to +\infty$; let $\gamma \in (0, \min(-\bar{\alpha}, \beta))$. It can be shown that there exist $c, d > 0$ such that $|y\,(t)| \le |y_1(t)| + cde^{-\gamma t}$ for t sufficiently large.

Lemma 2: Consider (A.1) where a belong to a *real* Ore field, $z(t)$ is real-valued and has a constant sign as $t \to +\infty$ and let $lim_{t\to+\infty} a(t) = \overline{a}$. If $\overline{a} < 0$ and $y(t)$ approaches zero exponentially when $t \to +\infty$, then $z(t)$ also approaches zero exponentially when $t \to +\infty$. $\diamond$

Sketch of the proof : As $y(t)$ approaches zero exponentially when $t \to +\infty$ and $\overline{a} < 0$, from (A.2) it follows that $y_2(t) = y(t) - y_1(t)$ also approaches zero exponentially when $t \to +\infty$. Moreover, y_2 can be written in the form

$$y_2(t) = \frac{f(t)}{g(t)},\ f(t) = \int ze^{-\int adt}dt,\ g(t) = e^{-\int adt} \tag{A.3}$$

where $e^{-\int adt} \to +\infty$ as $t \to +\infty$. Notice also that $\frac{df/dt}{dg/dt} = -\frac{z(t)}{a(t)}$. Since a belongs to a real Ore field and has a constant sign as $t \to +\infty$, f and g are comparable of order 1 near $+\infty$. $z(t)$ and $a(t)$ are also comparable. Thus, $lim_{t\to+\infty} y_2(t) = lim_{t\to+\infty} \frac{df/dt}{dg/dt} = -lim_{t\to+\infty} \frac{z(t)}{a(t)}$ from which the conclusion follows.

Let $Exp(a)$ denote the set of all $f \in \mathcal{O}_\infty$ such that $f : (B, +\infty) \to \mathbb{R}$, $f(t) \geq 0$ $(t \geq B)$ and $\chi(f) = a$, where $\chi(f) = limsup_{t\to+\infty} \frac{ln|f(t)|}{t}$ is the Lyapunov exponent of f.

Lemma 3:

(a) If $a \in \mathcal{O}_\infty$ is real-valued and is such that $lim_{t\to+\infty} a(t) = \bar{a} \in \bar{\mathbb{R}}$, then $e^{\int a(t)dt} \in Exp(\bar{a})$.

(b) If $f \in Exp(a_1)$ and $g \in Exp(a_2)$ $(a_1, a_2 \in \bar{\mathbb{R}})$, then

(i) $f + g \in Exp(a)$ where $a = sup(a_1, a_2)$;

(ii) $fg \in Exp(a_1 + a_2)$ where $(a_1, a_2) \neq (+\infty, -\infty)$.

(c) If $f \in Exp(a)$, then $\int f(t)dt \in Exp(a)$. $\diamond$

Sketch of the proof : Points (a) and (b) are obvious. For point (c), $ln(f(t)) \sim \bar{a}t$ (where $\bar{a}$ is finite). Thus, $ln(f(t)) = \bar{a}t + O(t) = \bar{a}(t + O(1))$ from which it follows that

$$f(t) = e^{\bar{a}(t+O(1))}, \tag{A.4}$$

which finally leads to $\chi(y) = \bar{a}$.

Lemma 4: Consider the differential equation

$$(\partial - a)^n y = 0, \quad a \in \mathcal{O}_\infty, \tag{A.5}$$

where n is a positive integer. Let $P(\partial) = (\partial - a)^n$ and consider the set $\mathcal{S}_P$ consisting of all solutions $y \in \mathcal{O}_\infty$ of $(A.5)$. Let $A \in \mathbb{R}$ be large enough, so that a, y can be viewed as analytic functions in $(A, +\infty)$ and let $B > A$. Consider $\overline{\alpha} = limsup_{t\to+\infty} \frac{\int_B^t \alpha(\tau)dt}{t-B}$, $\quad \underline{\alpha} = liminf_{t\to+\infty} \frac{\int_B^t \alpha(\tau)d\tau}{t-B}$ where $\alpha = Re\{a\}$.
(i) The set $\mathcal{S}_P$ is a $\mathbb{C}$-vector space of dimension n.
(ii) If $\overline{\alpha} < 0$, then all $y \in \mathcal{S}_P$ tend to zero exponentially as $t \to +\infty$.
(iii) If $\underline{\alpha} > 0$, then all solutions $y \in \mathcal{S}_P{}^{\times}$ are such that $|y(t)| \to +\infty$ exponentially as $t \to +\infty$.
(iv) If $\lim\limits_{t\to+\infty} \alpha(t) = 0$, then all solutions $y \in \mathcal{S}_P{}^{\times}$ are hypo-exponential. Moreover, they are bounded as $t \to +\infty$ if, and only if $n = 1$ and α is integrable on $[B, +\infty)$ for B large enough. These solutions do not tend to zero as $t \to +\infty$.

Sketch of the proof : (i) The germ a can be viewed as an analytic function in $(A, +\infty)$ for A large enough. Let $B > A$ and for any $i \in \{1, ..., n\}$, the general solution of $(A.5)$ in $[B, +\infty)$ is

$$y_n(t) = U(t, B) \sum_{0 \leq i \leq n-1} C_{n-i} \frac{(t-B)^i}{i!} \quad , \quad U(t, B) = e^{\int_B^t a(t)d\tau} \tag{A.6}$$

which proves (i).
(ii) If $\overline{\alpha} < 0$, then $U(t, B) \to 0$ exponentially as $t \to +\infty$, thus so does $|y_n(t)|$.
(iii) is similar to (ii).
(iv) is clear by the expression $(A.6)$.

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