INT. J. CONTROL, 1990, VOL. 52, NO. 5, 1217–1239

ρ-Stability and robustness: discrete-time case

H. BOURLÈS†, Y. JOANNIC‡ and O. MERCIER§

We say that a discrete-time system is ρ-stable if, roughly speaking, $\rho^k x_k \to 0$, where x_k is the system state. General ρ-stability theorems are established in this paper. They concern systems governed by functional difference equations. Systems of this type are encountered in the robustness studies. These ρ-stability theorems are a generalization of the well-known Lyapunov criterion. These results are applied to the robustness quantification problem in the second part of the paper. The case of discrete-time LQ regulators is deeply investigated. Robustness properties of continuous-time LQ regulators are found as the limit when the sampling period T tends to zero; robustness deteriorates as T increases. An upper bound is given for T, under which the robustness remains satisfactory. The practical interest of these theoretical results is illustrated on the basis of an industrial example.

1. Introduction

Stability and robustness are classical notions of control systems theory (Dorato 1987). The notion of α-stability, although classical for engineers, has been precisely defined and studied by Bourlès (1986 a, b, 1987 a) in the case of continuous-time systems. In this paper, where the case of discrete-time systems is considered, the above notion becomes the 'ρ-stability'. Roughly speaking, a discrete-time system is 'ρ-stable' if whatever the initial state x_0 is, $\rho^k x_k \to 0$, as $k \to \infty$ (where x_k denotes the state of the system at the normalized instant k). Typically, a linear time-invariant system, whose eigenvalues λj are such that $|\lambda j| < 1/\rho$, is ρ-stable.

Robustness has been broadly investigated for about 10 years, especially in the case of continuous-time LQ regulators (Anderson and Moore 1971, Safonov and Athans 1977). Recent and more complete results have been obtained by Bourlès (1986 a, b, 1987 a), where the α-stability of continuous-time systems, fed back by an LQ regulator, has been studied in the presence of modelling uncertainties.

Fewer results are available in the case of discrete-time systems, though important ones have been obtained by Safonov (1980). Preliminary work has also been made by Joannic (1983) and Joannic and Mercier (1983, 1986).

The main goal of this paper is to quantify explicitly the robustness properties of LQ regulators in the case of discrete time, as it has been done in the case of continuous time. The main contribution lies in the presentation of explicit conditions, allowing direct numerical computation of robustness bounds. These methods are illustrated on the basis of an industrial example.

The paper gives in § 2 some mathematical preliminaries; some useful formulae, which are straightforward generalizations of well-known results, are given without demonstration. Then new definitions are given concerning the notions of ρ-stability, ρ-detectability and ρ-gain. In § 3 general ρ-stability theorems are established. They

Received 11 April 1989. Revised 10 March 1990.
† EDF-DER/IMA/TIEM-1, avenue du Général de Gaulle, 92141 Clamart, France.
‡ MATRA-37, avenue Louis Breguet, B.P. 1, 78146 Velizy Villacoublay, France.

are based on the use of a Lyapunov function $V(x) = x^T \Pi x$ for a system governed by a functional difference equation, e.g. $x_{k+1} = (H * x)_k$—$*$ denoting the convolution product and H denoting an impulse response. Although it is not classical, this type of system is encountered in robustness studies. The approach developed in this section is a generalization of the classical Lyapunov theory. In § 4 the robustness problem is at first mathematically stated. Two kinds of uncertainties are taken into account: on the one hand, modelling uncertainties, adding to the right-hand member of the state equation. (For instance, let $x_{k+1} = Fx_k + Gu_k$ be the state equation; suppose that the coefficients of the matrix F are unprecisely known; then, this equation becomes

$$x_{k+1} = Fx_k + Gu_k + \Delta F x_k$$

where ΔF is the uncertainty on F; the function $x_k \to \Delta F x_k$ is the modelling uncertainty.) On the other hand, the control u which is computed is implemented via unmodelled actuators, and the state is measured via unmodelled filters and sensors. (This situation is very frequent in practice, because the order of the system becomes generally too large if all these devices are included into the model; on the other hand, some time constants of filters, and in certain cases of supplementary compensators such as PI, are adjusted only at the final step of the implementation.) The case of unmodelled actuators (or of unmodelled supplementary compensators) is more deeply investigated when the system is fed back by an LQ regulator. The results of Safonov (1980) are then generalized; indeed a ρ-stability is obtained (instead of an ordinary stability), and the two kinds of uncertainties mentioned above are taken into account simultaneously. In the case of sampled systems, the results obtained for continuous-time systems by Bourlès (1986 b, 1987 a) are found as the limit when the sampling period T tends to zero. Of course, robustness results deteriorate as T increases; a formula gives an upper bound for T, under which the robustness remains sufficient. Section 5 presents an application to the control of a turbogenerator unit. The influence of the sampling period is made obvious, along with the practical interest of the theoretical results. Section 6 is devoted to concluding remarks.

2. Mathematical preliminaries

2.1. *Notation and useful formulae*

ρ is a strictly positive number, denoting the stability degree imposed on the controlled system. This means that the control law is chosen such that the poles of the controlled system are strictly inside the circle of radius $1/\rho$ (in the linear case). The circle $|z| = 1/\rho$ of the complex plane is denoted by $C_{1/\rho}$.

Let E be a finite-dimensional normed space (e.g. $\mathbb{R}^n$); $\mathscr{C}(E)$ denotes the set of sequences $x = (x_k)$ of E such that $x_k = 0$ for $k < 0$ (causal sequences of E).

For $\rho = 1$ or 2, $l^p_\rho(E)$ is the subset of $\mathscr{C}(E)$ whose elements x are such that

$$\|x\|_{p,\rho} = \left(\sum_{k=0}^{\infty} \|\rho^k x_k\|^p \right)^{1/p} < \infty$$

Of course, $l^1_\rho(E) \subset l^2_\rho(E)$. For $p = 2$ and $E = \mathbb{R}^n$, $\langle x, y \rangle_\rho$ denotes the inner product

$$\langle x, y \rangle_\rho \triangleq \sum_{k=0}^{\infty} \rho^{2k} x_k^T y_k$$

$\mathscr{P}_N$ is the linear truncation operator (Willems 1971). It is defined for all N in $\mathbb{Z}$ by

$$(\mathscr{P}_N x)_k = x_k \text{ (respectively, 0)} \quad \text{if} \quad k \leqslant N \text{ (respectively, } k > N)$$

For x and y in $\mathscr{C}(E)$, we put

$$\|x\|_{p,\rho,N} \triangleq \|\mathscr{P}_N x\|_{p,\rho}$$

and

$$\langle x, y \rangle_{\rho,k} \triangleq \langle \mathscr{P}_k x, \mathscr{P}_k y \rangle_\rho$$

The definitions above are obvious generalizations of classical notions in input–output stability theory (Desoer and Vidyasagar 1975).

With this notation, the Parseval equality takes the form (for x and y in $l^2_\rho(E)$):

$$\langle x, y \rangle_\rho = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}^* \left(\frac{\exp(i\theta)}{\rho} \right) \hat{y} \left(\frac{\exp(i\theta)}{\rho} \right) d\theta$$

$$= \frac{1}{2\pi i} \int_{C_{1/\rho}} \hat{x}^*(z) \, \hat{y}(z) \, z^{-1} \, dz$$

where $\hat{x}$ is the z-transformation of x and $\hat{x}^*(z)$ is the transposed conjugate of $\hat{x}(z)$.

Let $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ be the set of linear mappings on $\mathbb{R}^n$ into $\mathbb{R}^m$, and $H \in \mathscr{C}(\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m))$, $K \in \mathscr{C}(\mathscr{L}(\mathbb{R}^m, \mathbb{R}^r))$, $x \in \mathscr{C}(\mathbb{R}^n)$. The convolution products $y = H * x$ and $L = K * H$ are obviously defined. Of course, one has $K*(H*x) = (K*H)*x$. Moreover, $\hat{y}(z) = \hat{H}(z)\,\hat{x}(z)$ and $\hat{L}(z) = \hat{K}(z)\,\hat{H}(z)$.

If $H \in l^1_\rho(\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m))$ and $x \in l^2_\rho(\mathbb{R}^n)$, then $y \in l^2_\rho(\mathbb{R}^m)$ and one proves easily that

$$\|H * x\|_{2,\rho} \leqslant \sup_{z \in C_{1/\rho}} \|\hat{H}(z)\| \cdot \|x\|_{2,\rho}$$

$$\leqslant \|H\|_{1,\rho} \|x\|_{2,\rho} \quad (1)$$

where $\|\hat{H}(z)\| = \bar{\sigma}(\hat{H}(z))$—the largest singular value of $\hat{H}(z)$—and

$$\|H\|_{1,\rho} = \sum_{k=0}^{\infty} \rho^k \bar{\sigma}(H_k)$$

If $K \in l^1_\rho(\mathscr{L}(\mathbb{R}^m, \mathbb{R}^r))$, then $L \in l^1_\rho(\mathscr{L}(\mathbb{R}^n, \mathbb{R}^r))$ and

$$\|L\|_{1,\rho} < \|K\|_{1,\rho} \|H\|_{1,\rho} \quad (2)$$

2.2. *ρ-Stability, ρ-detectability and ρ-gain*

In this section we define some generalizations of the classical notions of stability and detectability. For $\rho = 1$ they coincide with the classical notions. In the constant linear case, these generalizations are very obvious: a system is ρ-stable (respectively, ρ-detectable) if its poles (respectively, its unobservable modes) are strictly inside the circle $C_{1/\rho}$. These notions are now precisely defined in their complete generality.

Let there be the discrete-time system $\mathscr{S}$ defined by the equation

$$x_{k+1} = [\mathscr{F}(x)]_k + v_k, \quad k \geqslant 0 \quad (3)$$

where $\mathscr{F}$ is a causal operator on $\mathscr{C}(\mathbb{R}^n)$ into $\mathscr{C}(\mathbb{R}^n)$, and where $v \in \mathscr{C}(\mathbb{R}^n)$ denotes a perturbation. $\mathscr{S}$ will be interpreted later as the controlled system. (By 'causal' we mean that for all $k \in \mathbb{Z}$, $\mathscr{P}_k \mathscr{F} = \mathscr{P}_k \mathscr{F} \mathscr{P}_k$, i.e. at the instant k, $y_k = [\mathscr{F}(x)]_k$ is a function only of the past and present values $x_0, x_1, \ldots, x_k$ of x (Willems 1971).) If an initial value $x^0 \in \mathbb{R}^n$ is given, one proves easily by induction that (3) admits a unique solution $\varphi(x^0, v, \cdot): k \to \varphi(x^0, v, k)$, such that $\varphi(x^0, v, 0) = x^0$.

2.2.1. *ρ-Stability*

(*a*) *ρ-Stability in the sense of Lyapunov.* The point $0 \in \mathbb{R}^n$ is ρ-stable in the sense of Lyapunov (or LρS) for $\mathcal{S}$ if

$$\forall \varepsilon > 0, \quad \exists \eta > 0, \quad \|x^0\| < \eta$$
$$\Rightarrow \forall k \geqslant 0, \quad \|\rho^k \varphi(x^0, 0, k)\| < \varepsilon$$

The definition of other types of ρ-stability, such that the 'uniform Lρ-stability', where η does not depend on the initial time, is left to the reader.

(*b*) *ρ-Attractivity.* The point $0 \in \mathbb{R}^n$ is ρ-attractive (respectively, globally ρ-attractive) for $\mathcal{S}$ if there exists a neighbourhood U of 0 in $\mathbb{R}^n$ such that

$$\forall x^0 \in U \text{ (respectively, } \forall x^0 \in \mathbb{R}^n) \lim_{k \to \infty} \rho^k \varphi(x^0, 0, k) = 0$$

(*c*) *Asymptotic ρ-stability.* The point $0 \in \mathbb{R}^n$ is asymptotically ρ-stable (AρS) (respectively, globally asymptotically ρ-stable—(GAρS) for $\mathcal{S}$ if

(i) 0 is LρS for $\mathcal{S}$;
(ii) 0 is ρ-attractive (respectively, globally ρ-attractive) for $\mathcal{S}$.

For convenience, if 0 is LρS (respectively, AρS or GAρS) for $\mathcal{S}$, we say in the following, by some abuse of language, that $\mathcal{S}$ is LρS (respectively, AρS or GAρS).

(*d*) *ρ-Stability in the input–output sense.* $\mathcal{S}$ is l^p-ρ-stable in the input–output sense (IO $l^p\rho$S) if there exist strictly positive numbers γ and δ such that

$$\forall x^0 \in \mathbb{R}^n, \quad \forall v \in \mathscr{C}(\mathbb{R}^n), \quad \forall k \geqslant 0, \quad \|\varphi(x^0, v, \cdot)\|_{p,\rho,k} \leqslant \gamma + \delta \|v\|_{p,\rho,k}$$

In the following, only the case $p = 2$ will be studied, and instead of the IO-l^2-ρ-stability, we speak more concisely of the IO-ρ-stability (IOρS).

2.2.2. *ρ-Detectability.* Let $C \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$, $m \leqslant n$, so that the sequence $y_k = Cx_k$ can be interpreted as the output of $\mathcal{S}$.

(*a*) The pair $(C, \mathcal{S})$ is ρ-detectable if

$$\forall x^0 \in \mathbb{R}^n, \quad (C\varphi(x^0, 0, \cdot) \in l_\rho^2(\mathbb{R}^m) \Rightarrow \varphi(x^0, 0, \cdot) \in l_\rho^2(\mathbb{R}^n))$$

This type of detectability was first defined by Safonov (1977) for continuous-time systems, and was then extended by Bourlès (1986 a), where the notion of α-detectability was introducted. The case $\rho = 1$ was considered by Joannic (1983 a) and Joannic and Mercier (1983) in the discrete-time case.

(*b*) The pair $(C, \mathcal{S})$ is strongly ρ-detectable if

(i) $(C, \mathcal{S})$ is ρ-detectable
(ii) $\forall \varepsilon > 0, \quad \exists \eta > 0, \quad \forall x^0 \in \mathbb{R}^n, \quad \forall k \geqslant 0$
$$\|C\varphi(x^0, 0, \cdot)\|_{2,\rho,k} \leqslant \eta \Rightarrow \|\varphi(x^0, 0, \cdot)\|_{2,\rho,k} \leqslant \varepsilon$$

The notion of strong detectability has been defined by Bourlès (1984) for continuous-time systems, and generalized by Bourlès (1986 a), where the notion of strong α-detectability is introduced. By the same reasoning as of Bourlès (1984, 1986 a) one obtains the following proposition.

Proposition 1

(*a*) Suppose (3) reduces to

$$x_{k+1} = Ax_k + v_k$$

where $A \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then

(i) $(C, \mathcal{S})$ is ρ-detectable iff, in the classical sense, the pair $(C, \rho A)$ is detectable
(ii) $(C, \mathcal{S})$ is strongly ρ-detectable iff the pair (C, A) is observable

(*b*) If $m = n$ and C is invertible, then $(C, \mathcal{S})$ is strongly ρ-detectable (for the general equation (3)).

2.2.3. *ρ-Gain.* The gain of an operator is a classical notion given by Zames (1966), Willems (1971) and Vidyasagar (1981). We give here the discrete-time version of the generalization made by Bourlès (1986 a, b).

Let $\mathscr{F}$ be an operator on $\mathscr{C}(\mathbb{R}^n)$ into $\mathscr{C}(\mathbb{R}^n)$; the ρ-gain of $\mathscr{F}$ is the quantity

$$\gamma_\rho(\mathscr{F}) \triangleq \inf \{a > 0, \exists b \geqslant 0 : \forall x \in \mathscr{C}(\mathbb{R}^n), \quad \forall k \in N,$$
$$\|\mathscr{F}(x)\|_{2,\rho,k} \leqslant a\|x\|_{2,\rho,k} + b\}$$

The ρ-gain of $\mathscr{F}$ with zero bias is the quantity

$$\gamma_\rho^0(\mathscr{F}) \triangleq \inf \{a > 0, \quad \forall x \in \mathscr{C}(\mathbb{R}^n), \quad \forall k \in N,$$
$$\|\mathscr{F}(x)\|_{2,\rho,k} \leqslant a\|x\|_{2,\rho,k}\}$$

If $\gamma_\rho(\mathscr{F}) < \infty$ (respectively, $\gamma_\rho^0(\mathscr{F}) < \infty$), we say that $\mathscr{F}$ has a finite ρ-gain (respectively, a finite ρ-gain with zero bias). Of course, one has

$$0 \leqslant \gamma_\rho(\mathscr{F}) \leqslant \gamma_\rho^0(\mathscr{F}) \leqslant \infty \quad \text{and} \quad \gamma_\rho^0(\mathscr{F}) = \sup_{\substack{x \in \mathscr{C}(\mathbb{R}^n) \\ k \in N}} \frac{\|\mathscr{F}(x)\|_{2,\rho,k}}{\|x\|_{2,\rho,k}}$$

If $\mathscr{F}$ is linear then $\gamma_\rho(\mathscr{F}) = \gamma_\rho^0(\mathscr{F})$.

Note that $\gamma_\rho^0(\lambda\mathscr{F}) = |\lambda|\gamma_\rho^0(\mathscr{F})$ for all $\lambda \in \mathbb{R}$, and that

$$\gamma_\rho^0(\mathscr{F}_1 + \mathscr{F}_2) \leqslant \gamma_\rho^0(\mathscr{F}_1) + \gamma_\rho^0(\mathscr{F}_2)$$

Moreover, $\gamma_\rho^0(\mathscr{F}) = 0$, iff $\mathscr{F} = 0$, so that γ_ρ^0 is a norm on the vector space of linear operators $\mathscr{F}$ for which $\gamma_\rho^0(\mathscr{F}) < \infty$.

Example 1

Let $\mathscr{F}$ be such that $\mathscr{F}(x) = H * x$, where $H \in l_\rho^1(\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n))$. Then by (1)

$$\gamma_\rho^0(\mathscr{F}) = \sup_{z \in C_{1/\rho}} \|\hat{H}(z)\| \leqslant \|H\|_{1,\rho} < \infty$$

Note that for $\rho = 1$, one has

$$\sup_{z \in C_1} \|\hat{H}(z)\| = \|\hat{H}\|_\infty$$

where $\|\hat{H}\|_\infty$ is the norm of the transfer matrix H in the appropriate Hardy space $\mathscr{H}_\infty$, i.e.

$$\|\hat{H}\|_\infty = \sup_{|z|\geq 1} \|\hat{H}(z)\|$$

Example 2

Let $\mathscr{F}$ be such that $\forall\, k \geq 0$ and $\forall\, x \in \mathscr{C}(\mathbb{R}^n)$, $[\mathscr{F}(x)]_k = f_k(x)$, where f_k is a sequence of functions on $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfying the condition

$$\exists\, \mu \geq 0, \quad \forall\, k \geq 0, \quad \forall\, \xi \in \mathbb{R}^n, \quad \|f_k(\xi)\| \leq \mu\|\xi\|$$

Then $\gamma_\rho^0(\mathscr{F}) \leq \mu$.

3. ρ-Stability theorems

In this section we give sufficient conditions for the system $\mathscr{S}$ to be ρ-stable.

3.1. *Non-linear case*

The following theorem applies to the general case where $\mathscr{F}$ is a non-linear operator. It is proved in Appendix A and can be interpreted as a generalization of the Lyapunov criterion, as shown later.

Theorem 1

Suppose there exist two symmetric positive (≥ 0) operators Π and Σ in $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that $\forall\, x \in \mathscr{C}(\mathbb{R}^n)$, $\forall\, k \in N$

$$\rho^2 \langle \mathscr{F}(x), \quad \Pi\mathscr{F}(x)\rangle_{\rho,k} \leq \langle x, (\Pi - \Sigma)x\rangle_{\rho,k} \tag{4}$$

Let $\Sigma^{1/2}$ denote the symmetric positive square root of Σ, then we have the following.

(i) If $(\Sigma^{1/2}, \mathscr{S})$ is ρ-detectable, then 0 is globally ρ-attractive for $\mathscr{S}$.

(ii) If $(\Sigma^{1/2}, \mathscr{S})$ is strongly ρ-detectable, then $\mathscr{S}$ is GAρS.

(iii) If Σ is positive definite (> 0) and $\gamma_\rho(\mathscr{F}) < \infty$, then $\mathscr{S}$ is GAρS and IOρS.

Case of a perturbated system. Now suppose that $\mathscr{F} = \mathscr{F}_1 + \mathscr{F}_2$, where

(*a*) $\mathscr{F}_1$ satisfies a condition similar to (4) with $\Sigma > 0$ (so that $\mathscr{S}$ is ρ-stable for $\mathscr{F}_2 = 0$)

(*b*) $\mathscr{F}_2$ is a 'perturbation operator' such that $\gamma_\rho^0(\mathscr{F}_2)$ is 'sufficiently small'

In this situation, one obtains the following theorem, proved in Appendix B.

Theorem 2

Suppose that there exist two symmetric operators Σ and Π in $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$, satisfying $0 < \Sigma \leq \Pi$ (i.e. Σ is positive definite and $\Pi - \Sigma$ is positive), such that

$$\forall\, x \in \mathscr{C}(\mathbb{R}^n), \quad \forall\, k \in \mathbb{N}$$

$$\rho^2 \langle \mathscr{F}_1(x), \Pi\mathscr{F}_2(x)\rangle_{\rho,k} \leq \langle x, (\Pi - \Sigma)x\rangle_{\rho,k}$$

$$\gamma_\rho^0(\mathscr{F}_2) < \frac{1}{2\rho}\frac{\underline{\sigma}(\Sigma)}{\bar{\sigma}(\Pi)} \tag{5}$$

where $\underline{\sigma}$ (respectively, $\bar{\sigma}$) denotes the smaller (respectively, larger) singular value. Then $\mathscr{S}$ is GAρS and IOρS.

3.2. *Constant linear case*

Now let us consider the case where $\mathscr{F}_1$ is a convolution operator (see Example 1). Let $\hat{H}(z)$ be the transfer matrix of $\mathscr{F}_1$. One has the following proposition, proved in Appendix C.

Proposition 2

If there exists symmetric positive operators H and U in $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\forall\, z \in C_{1/\rho}, \quad \hat{H}^*(z)\Pi\hat{H}(z) \leq U \tag{6}$$

then $\forall\, k \in \mathbb{N}$ and $\forall\, x \in \mathscr{C}(\mathbb{R}^n)$

$$\langle \mathscr{F}_1(x), \Pi\mathscr{F}_1(x)\rangle_{\rho,k} \leq \langle x, Ux\rangle_{\rho,k}$$

Note that in (6), U is extended on $\mathbb{C}^n$ by

$$U(\alpha + i\beta) = U(\alpha) + iU(\beta)$$

where $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^n$ and $U(\alpha + i\beta) \in \mathbb{C}^n$.

By Theorem 2 and Proposition 2, one obtains the following theorem.

Theorem 3

Suppose that there exist two symmetric operators Π and Σ in $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$, satisfying $0 < \Sigma \leq \Pi$, such that

(i) $\forall\, z \in C_{1/\rho}, \quad \rho^2\hat{H}^*(z)\Pi\hat{H}(z) \leq \Pi - \Sigma$

(ii) the condition (5) is satisfied

Then, $\mathscr{S}$ is GAρS and IOρS.

Remarks

This result is the discrete-time version of Theorem 3 of Bourlès (1986 b).

Let us consider the case $\mathscr{F}_2 = 0$. Then, the conditions above reduce to (i); this condition is a generalization of Theorem 1 of Joannic (1983), and Joannic and Mercier (1983), where the case $\rho = 1$ was considered. Moreover, suppose that $H_k = A\delta_k$, where $A \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ and where δ is the unit impulse. Then in the absence of perturbation v, the equation (3) of $\mathscr{S}$ becomes

$$x_{k+1} = Ax_k$$

and (*i*) reduces to

$$\rho^2 A^{\mathrm{T}}\Pi A \leq \Pi - \Sigma$$

which is the sufficient condition of the Lyapunov criterion (for $\rho = 1$).

4. Robustness of discrete-time regulators

In this section, the preceding results are applied to the robustness quantification problem. This problem has been mathematically stated for continuous-time systems by Bourlès (1982, 1984, 1986 a) and for discrete-time systems by Joannic (1983), and Joannic and Mercier (1983). This statement is now recalled for the convenience of the reader.

4.1. *Mathematical statement of the robustness problem and general robustness theorem*

We consider the problem of controlling a discrete-time system $\mathscr{S}_0$. Suppose that this system is modelled by the linear equation

$$x_{k+1} = Fx_k + Gu_k \tag{7}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $F \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $G \in \mathscr{L}(\mathbb{R}^m, \mathbb{R}^n)$

Because all such models are approximations, suppose that $\mathscr{S}_0$ is governed in reality by the more general equation

$$x_{k+1} = Fx_k + Gu_k + [\mathscr{F}_0(x, u)]_k + v_k \tag{8}$$

where $\mathscr{F}_0(x, u)$ is the *model error* and v is an *external perturbation*. In the following, it is supposed that $\mathscr{F}_0$ is a causal operator on $\mathscr{C}(\mathbb{R}^n) \times \mathscr{C}(\mathbb{R}^m)$ into $\mathscr{C}(\mathbb{R}^n)$—i.e. $\forall k$, $\forall (x, u)$, $\mathscr{P}_k \mathscr{F}_0(x, u) = \mathscr{P}_k \mathscr{F}_0(\mathscr{P}_k x, \mathscr{P}_k u)$—and that $v \in \mathscr{C}(\mathbb{R}^n)$.

The objective is to stabilize $\mathscr{S}_0$ by a linear control u, i.e.

$$u_k = Kx_k \tag{9}$$

where $K \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ is the gain of our regulator.

Consider now the implementation of the control action (9). Often, transfer functions of some sensors, filters and actuators have been neglected (to avoid increasing the dimension of the system state). Therefore, in a realistic way the feedback loop takes the form illustrated in Fig. 1, where $\hat{L}(z)$ (respectively, $\hat{M}(z)$) denotes the transfer matrix of the unmodelled actuators (respectively, filters).

The equations of the feedback loop can be expressed in the time domain as

$$x_{k+1} = Fx_k + Gu'_k + [\mathscr{F}_0(x, u')]_k + v_k$$
$$u'_k = (L * u)_k$$
$$u_k = Kx'_k$$
$$x'_k = (M * x)_k$$

and finally the closed-loop equation is

$$x_{k+1} = [\mathscr{F}_1(x) + \mathscr{F}_2(x)]_k + v_k \tag{10}$$

where

$$\mathscr{F}_1(x) = (F\delta + GL * KM) * x \triangleq H * x \tag{11}$$

$$\mathscr{F}_2(x) = \mathscr{F}_0(x, L * KM * x) \tag{12}$$

The robustness quantification problem now can be mathematically stated as follows. Let K be the gain of a regulator for which the closed-loop system $\mathscr{S}$ is ρ-stable (or more precisely GAρS and IOρS) with $\mathscr{F}_0 = 0$, $\hat{L}(z) = I_m$, $\hat{M}(z) = I_n$ (i.e. no model error, and perfect actuators, sensors and filters). For which condition about $\mathscr{F}_0$, $\hat{L}(z)$ and $\hat{M}(z)$, does $\mathscr{S}$ remain ρ-stable when $\mathscr{F}_0$, $\Delta\hat{L}(z) \triangleq \hat{L}(z) - I_m$ and $\Delta\hat{M}(z) \triangleq \hat{M}(z) - I_n$ are no longer equal to 0?

The condition about $\mathscr{F}_0$ expresses the 'structural robustness' of the regulator, and

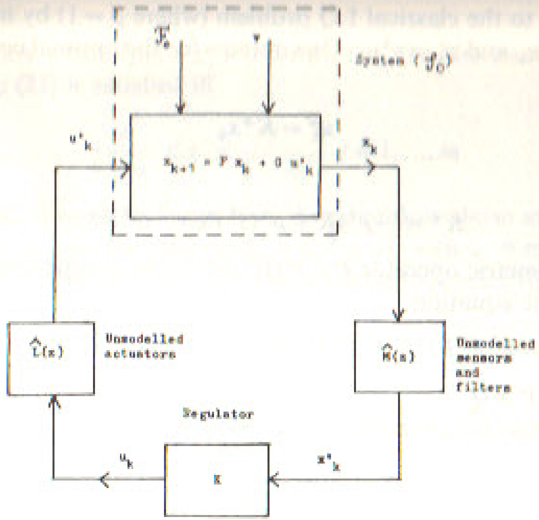

Figure 1. Perturbed controlled system.

the condition about $\hat{L}(z)$ and $\hat{M}(z)$ expresses its 'stability margin'. Of course, for the existence of K we have to assume that $(\rho F, \rho G)$ is stabilizable.

The following theorem is an immediate consequence of Theorem 3.

Theorem 4

Assume that there exist two symmetric operators Π and Σ in $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$, satisfying $0 < \Sigma \leqslant \Pi$, and such that

$$\forall z \in C_{1/\rho}, \quad \rho^2(F + G\hat{L}(z)K\hat{M}(z))^*\Pi(F + G\hat{L}(z)K\hat{M}(z)) \leqslant \Pi - \Sigma \tag{13}$$

$$\gamma_\rho^0(\mathscr{F}_2) < \frac{1}{2\rho}\frac{\underline{\sigma}(\Sigma)}{\bar{\sigma}(\Pi)}$$

Then $\mathscr{S}$ remains GAρS and IOρS.

By the Lyapunov criterion, there exists a symmetric operator $\Pi > 0$ such that

$$\rho^2(F + GK)^{\mathrm{T}}\Pi(F + GK) = \Pi - I_n$$

It follows by a continuity argument that there exists $\Sigma > 0$ such that (13) is satisfied if $\|\Delta\hat{L}(z)\|$ and $\|\Delta\hat{M}(z)\|$ are sufficiently small for all $z \in C_{1/\rho}$. By (1), it suffices that $\|\Delta L\|_{1,\rho}$ and $\|\Delta M\|_{1,\rho}$ be sufficiently small. This expresses the fact that GA-ρ-stability and the IO ρ-stability are generic notions. In the general case, however, the bounds about $\|\Delta L\|_{1,\rho}$ and $\|\Delta M\|_{1,\rho}$ are complicated and unusable in practice. For this reason, the particular case of an LQ regulator is studied in the next section, where simple and useful conditions are obtained.

4.2. *Robustness of discrete-time LQ regulators*

4.2.1. *Definition of the regulator*. Let be the system (7) and the quadratic index

$$J = \sum_{k=0}^{\infty} \rho^{2k}(x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k) \tag{14}$$

Where $(\rho F, \rho G)$ is stabilizable, and $Q > 0$, and $R > 0$.

One can return to the classical LQ problem (where $\rho = 1$) by making the change of variables $x'_k = \rho^k x_k$ and $u'_k = \rho^k u_k$. One obtains for the optimal control u^*, minimizing (14) under (7)

$$u_k^* = K^* x_k$$

with

$$K^* = -\rho^2 (R + \rho^2 G^T P G)^{-1} G^T P F \tag{15}$$

and where the symmetric operator $P \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ is the unique positive definite solution of the algebraic equation

$$P = \rho^2 F^T P F - \rho^4 F^T P G (R + \rho^2 G^T P G)^{-1} G^T P F + Q \tag{16}$$

Note that by (16), $P > Q$.

Finally, we choose for controlling the system $\mathscr{S}_0$ the gain K defined by

$$K = \frac{1}{\gamma} K^* \tag{17}$$

where $\gamma > 0$ is a scale function which will be useful in the following. One such adjustable parameter was used by Bourlès (1986 a, 1987 a) in the continuous-time case. Some restrictions will be imposed to γ in the next section. (Of course, the stability may be lost for certain values of γ.)

4.2.2. *Robustness results.* Assume for convenience that $\hat{M}(z) = I_n$ (i.e. we look at the stability margin only at the point where the control applies) and that for all $z \in C_{1/\rho}$, $\hat{L}(z)$ is invertible. Of course, this last assumption is satisfied if $\|\Delta L\|_{1,\rho} < 1$, i.e. if the perturbation due to the presence of unmodelled actuators is not too large. With these assumptions, one obtains by applying Theorem 3 (with $\Pi = P$ and $\Sigma = Q$) the following theorem, proved in Appendix D.

Theorem 5

$\mathscr{S}$ remains GAρS and IOρS if

(i) $\forall z \in C_{1/\rho}$

$$[I_m - \gamma \hat{L}(z)^{-1}]^* (R + \rho^2 G^T P G)[I_m - \gamma \hat{L}(z)^{-1}] \leqslant R \tag{18}$$

(ii)

$$\gamma_\rho^0(\mathscr{F}_2) \leqslant \frac{1}{2\rho} \frac{\underline{\sigma}(Q)}{\bar{\sigma}(P)} \tag{19}$$

If $R = \mathrm{diag}(r_i)$ and $\hat{L}(z) = \mathrm{diag}(\hat{l}_i(z))$, the condition (i) may be replaced by

$$\sup_{\substack{z \in C_{1/\rho} \\ j=1,\dots,m}} \left| 1 - \frac{\gamma}{\hat{l}_j(z)} \right| \leqslant a_j \triangleq \left(\frac{r_j}{r_j + \rho^2 \sigma(G^T P G)} \right)^{1/2} \tag{20}$$

or, in an equivalent manner, by

$$\sup_{\substack{z \in C_{1/\rho} \\ j=1,\dots,m}} \left| \hat{l}_j(z) - \frac{\gamma}{1 - a_j^2} \right| \leqslant \frac{\gamma a_j}{1 - a_j^2} \tag{21}$$

For $\rho = \gamma = 1$ and $\mathscr{F}_2 = 0$, the above inequality coincides with the result obtained by Safonov (1980)—see Corollary 4.1 and Figure 4.5.

Gain margin

Consider the case where $\hat{l}_j(z) = g_j > 0$ for $z \in C_{1/\rho}$ (i.e. the transfer function $\hat{l}_j(z)$ is a pure gain). Then (21) is satisfied iff

$$\frac{\gamma}{1 + a_j} \leqslant g_j \leqslant \frac{\gamma}{1 - a_j}, \quad j = 1, \dots, m$$

Of course, in the ideal case (i.e. for $g_j = 1$), the condition above must be satisfied, so that one must choose γ in the interval $[1 - a, 1 + a]$, with $a \triangleq \min\{a_j, j = 1, \dots, m\}$. For $m = 1$ (monovariable case), the condition above expresses that the *gain margin* of our regulator is

$$\left[\frac{\gamma}{1 + a}, \frac{\gamma}{1 - a} \right]$$

Phase margin

Assume that $\gamma = 1$ and $\hat{l}_j(z) = \exp(i\varphi_j)$, $-\pi < \varphi_j \leqslant \pi$ (pure phase shift of angle φ_j). Then (20) is satisfied iff

$$|\varphi_j| \leqslant 2 \arcsin \frac{a_j}{2}, \quad j = 1, \dots, m$$

This result has been obtained by Safonov (1980) in the case $\rho = 1$.

Geometric interpretation of conditions (20) and (21) and optimal choice of γ for obtaining the maximal phase margin

Let $C(\Omega, r)$ be the circle of centre Ω ($\Omega \in \mathbb{C}$) and radius r, and let J_0 be the transformation defined by $J_0(z) = z^{-1}$. Let also $\hat{C}_j(z) \triangleq \hat{l}_j(z)/\gamma$. By (20), (21),

$$J_0(C(1, a)) = C\left(\frac{1}{1 - a^2}, \frac{a}{1 - a^2} \right) \triangleq \Gamma(a)$$

and a sufficient condition for (20) to be satisfied is

$$\forall z \in C_{1/\rho}, \ \hat{C}_j(z) \in \Delta(a), \quad j = 1, \dots, m \tag{22}$$

where $\Delta(a)$ denotes the closed disc inside $\Gamma(a)$; this condition is necessary and sufficient for $R = rI_m$.

Because the circles $\Gamma(a)$ were first used by Safonov (1980), they may be called 'Safonov's circles'. Some of them are represented in Fig. 2. Note that $\Gamma(a)$ may be obtained from the M-circle (with $M = 1/a$) of the classical control theory, by a symmetry with respect to the imaginary axis.

As $\Delta(a) \subset \Delta(a')$ for $a < a'$, the greater is a, the less binding is condition (22). As $a < 1$, the more favourable case is obtained for a tending to 1, and then (22) becomes $\mathrm{Re}[\hat{C}_j(z)] \geqslant 1/2$. It will be seen later that this result is of a great practical interest in the case of sampled system.

Some elementary geometrical considerations prove that the quantities Φ, Φ^* and γ^* of Fig. 2 are given by

$$\Phi = 2 \arcsin\left(\frac{a}{2} \right) \tag{23}$$

$$\gamma^* = (1 - a^2)^{1/2} \tag{24}$$

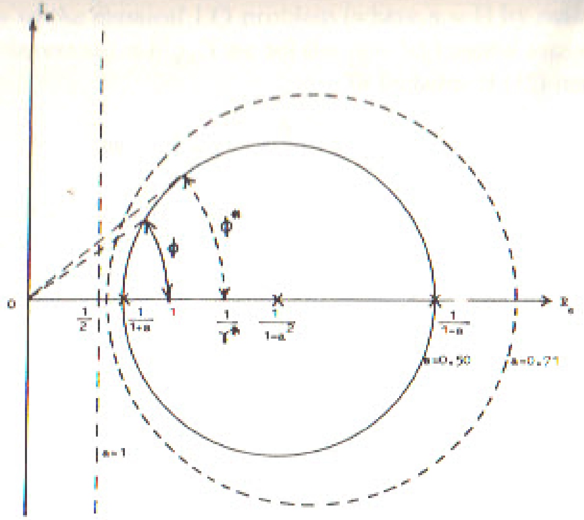

Figure 2. Safonov's circles.

$$\Phi^* = \arcsin a > \Phi \tag{25}$$

Note that $\hat{l}_j(z) = \exp(i\varphi_j)$ iff $\hat{C}_j(z) = 1/\gamma \exp(i\varphi_j)$, i.e. at a pure phase shift φ_j corresponds a rotation of $\hat{C}_j(z)$ of the angle φ_j on the circle $C(0, 1/\gamma)$ in Fig. 2.

For $\gamma = 1$, this angle φ_j must verify $|\varphi_j| \leqslant \Phi$ for the condition (22) to be satisfied, so that Φ is the phase margin, as shown above. Clearly, the maximum angle allowed is Φ^*, corresponding to $\gamma = \gamma^*$. Therefore, γ^* is the value of γ for which the maximum phase margin, i.e. Φ^*, is obtained.

Interpretation and use of condition (19)

Assume for convenience that the structural perturbation $\mathscr{F}_0$ consists of an error ΔF on the state matrix F, so that $\gamma_\rho^0(\mathscr{F}_2) = \bar{\sigma}(\Delta F)$. Therefore, the condition (19) becomes

$$\bar{\sigma}(\Delta F) \leqslant \frac{1}{2\rho} \frac{\underline{\sigma}(Q)}{\bar{\sigma}(P)} \triangleq \mu \tag{26}$$

Of course this condition is satisfied if the error ΔF is 'small'. However, the larger μ, the larger the set of 'admissible errors' ΔF, i.e. those verifying (26). Therefore, a structurally robust regulator is obtained if the cost matrices Q and R are chosen in such a manner that the quantity μ defined above is sufficiently large. A steepest descent algorithm can be used for increasing μ systematically (Bourlès 1982), (Mercier *et al.* 1983).

4.3. *Case of sampling systems*

Now let us consider the case where $\mathscr{S}_0$ has been obtained by zero-order-hold sampling at the period T of the continuous-time system.

$$\dot{x}_c = Ax_c + Bu_c \tag{27}$$

It is well known that the sampled system is of the form (7) with $x_k = x_c(kT)$, $u_k = u_c(kT)$ and

$$F = \exp(AT), \quad G = \int_0^T \exp(A\tau)\, d\tau\, B \tag{28}$$

Approximations

By (28), one has at the neighbourhood of $T = 0$

$$F = I + TA + o(T) \tag{29}$$

$$G = TB + o(T) \tag{30}$$

Now assume that the discrete-time regulator defined in § 4.3.1 is used for controlling the system. This regulator generates a discrete control u_k which is an approximation of a continuous control u_c. The robustness properties of u_c have been quantified by Bourlès (1986 b, 1987 a). This control u_c is obtained in the following manner.

Let $u_c^* = K_c^* x_c$ be the control which minimizes under the dynamical constraint (27) the quadratic index

$$J_c = \int_0^\infty \exp(2\alpha t)\,(x_c^T Q_c x_c + u_c^T R_c u_c)\, dt \tag{31}$$

where $\alpha \geqslant 0$, $Q_c > 0$, $R_c > 0$. (For the existence of u_c^*, we assume that $(A + \alpha I, B)$ is stabilizable.) Then u_c is defined by

$$u_c = K_c x_c \tag{32}$$

where

$$K_c = \frac{1}{\gamma} K_c^*$$

If the integral J_c and the series J are convergent, one has $J \sim J_c$ at the neighbourhood of $T = 0$, where J is given by (14) with

$$Q = TQ_c \tag{33}$$

$$R = TR_c \tag{34}$$

$$\rho = \exp(\alpha T) \tag{35}$$

Therefore, for $T \to 0$, minimizing J_c for (27) is the same as minimizing J for (7).

This proves well, as said above, that u (with the zero-order hold) converges to u_c when $T \to 0$. This can also be proved directly by taking the limits of (15) and (16) when $T \to 0$. Indeed, it is easily proved that P converges to the positive definite solution P_c of the algebraic equation (corresponding to the continuous-time case)

$$(A + \alpha I)^T P_c + P_c(A + \alpha I) - P_c B R_c^{-1} B^T P_c + Q_c = 0$$

Moreover, K^* converges to K_c^* defined by

$$K_c^* = -R_c^{-1} B^T P_c$$

which is the optimal gain for J_c.

Stability margin

By the preceeding results, one has $\rho^2 G^T PG = o(T)$ and $R \sim TR_c$ at the neighbourhood of $T = 0$, so that the number a defined above converges to 1 when $T \to 0$. Moreover, when a approaches 1, $\Delta(a)$ becomes the closed half-plane $\mathrm{Re}\,(z) \geqslant \frac{1}{2}$, so that condition (22) becomes

$$\inf_{\substack{z \in C_{1/\rho} \\ i=1,\ldots,m}} \mathrm{Re}\,[\hat{l}_j(z)] \geqslant \frac{\gamma}{2} \tag{36}$$

Now let $\hat{L}_a(s) = \mathrm{diag}\,(\hat{l}_{a_j}(s))$, the continuous-time transfer matrix of the actuators. For small values of T, the discretization can be made, for instance, by the Euler method:

$$s = \frac{z-1}{T} \tag{37}$$

All z in $C_{1/\rho}$ can be written in the form

$$z = \frac{1}{\rho}\exp(i\omega) = \exp[(i\Omega - \alpha)T]$$

with ρ verifying (35) and $\Omega \triangleq \omega T$. Hence by (37) one has at the neighbourhood of $T = 0$

$$s = -\alpha + i\Omega + o(1)$$

so that for $T \to 0$, (36) becomes

$$\inf_{\substack{\mathrm{Re}(s) = -\alpha \\ i=1,\ldots,m}} \mathrm{Re}\,[\hat{l}_{aj}(s)] \geqslant \frac{\gamma}{2} \tag{38}$$

We recapture here the stability margin result obtained by Bourlès (1986 a, b) in the continuous-time case. In particular, the gain margin become equal to $[\gamma/2, \infty[$ and for $\gamma = 1$ the phase margin becomes equal to 60°.

Note that the optimal quantities γ^* and Φ^* defined by (24) and (25) become, respectively, 0 and $\pi/2$, so that they are of no interest here.

Structural robustness

Assume that an error ΔA has been made on the state matrix A. By (29), at this error ΔA corresponds an error ΔF on F, such that $\Delta F = T\Delta A + o(T)$. Therefore, for $T \to 0$, the condition (26) becomes

$$\sigma(\Delta A) \leqslant \frac{\underline{\sigma}(Q_c)}{2\bar{\sigma}(P_c)}$$

This is the structural robustness property obtained by Bourlès (1986 a, 1987) in the continuous-time case.

Choice of the sampling period

The results above show that the robustness properties of continuous-time LQ regulators—in particular the stability margin expressed by the condition (38) is degraded when the discrete-time approximation is used. The shorter the sampling period, the less the difference between the two situations, so that there necessarily exists an upper bound T^*, such that for $T \leqslant T^*$ this difference remains tolerable.

By substituting (33 and 34) into (20), one obtains at the neighbourhood of $T = 0$

$$a_j = 1 - \frac{T}{2}\frac{\bar{\sigma}(B^T P_c B)}{r_{cj}} + o(T)$$

so that

$$a = 1 - \frac{T}{2}\frac{\bar{\sigma}(B^T P_c B)}{\underline{\sigma}(R_c)} + o(T) \tag{39}$$

For $\gamma = 1$, it has been proved that the phase margin is equal to 60° (i.e. $\pi/3$ rad) in the continuous-time case, and to Φ given by (23) in the discrete-time case. Assume that we want to obtain $\Phi \geqslant \pi/3 - \Delta\Phi$, where $\Delta\Phi$ is the difference judged as tolerable between the two situations (for instance, $\Delta\Phi = 10°$, i.e. $\pi/18$ rad). By (23), one has at the neighbourhood of $\Delta\Phi = 0$

$$a = 1 - \frac{\sqrt{3}}{2}\Delta\Phi + o(\Delta\Phi)$$

Therefore, one obtains by (39)

$$T^* \simeq \sqrt{3}\,\frac{\underline{\sigma}(R_c)}{\bar{\sigma}(B^T P_c B)}\Delta\Phi \tag{40}$$

For example, let us consider the first-order system

$$\tau\dot{x} = x - u$$

The solution of the algebraic 'Riccati equation' corresponding to the optimization problem (31)—with $\alpha = 0$, $Q_c = q > 0$, $R_c = r > 0$—is given by

$$P_c = \tau r[1 + (1 + q/r)^{1/2}]$$

so that (40) becomes

$$T^* = \tau\frac{\sqrt{3}}{1 + \sqrt{1 + q/r}}\Delta\Phi$$

For $q = r$ and $\Delta\Phi = \pi/18$, one obtains $T^* \simeq 0{\cdot}125\tau$.

5. Application to the regulation of a turbo-generator unit

In this section we apply the above results to the regulation of a turbo-generator unit. A simplified third-order linear model is considered (Bénéjean 1986, Irving *et al.* 1988).

The state x_c of the continuous system is

$$x_c = (\Delta U_s/U_{sn}, \Delta\Omega/\Omega_n, \Delta P_e/P_{e_n})$$

where U_s is the terminal voltage, Ω is the rotation speed, P_e is the electric active power delivered by the machine, Δ denotes the difference with the nominal value, and the subscript n denotes the nominal value (i.e. P_{e_n} is the nominal value of P_e, and $\Delta P_e = P_e - P_{e_n}$).

The input of the system is

$$u_c = \frac{\Delta U_a}{U_{an}}$$

where U_a is the excitor input voltage, and the output is

$$y_c = \frac{\Delta P_e}{P_{e_0}} = (0 \quad 0 \quad 1)x_c$$

For the linearization point considered in this application, the linear model is of the form (27) with

$$A = \begin{bmatrix} -0{\cdot}8650 & -5{\cdot}3206 & -0{\cdot}2476 \\ 0 & -1{\cdot}124 & -2{\cdot}0833 \\ -2{\cdot}0020 & 0{\cdot}8533 & 0{\cdot}4352 \end{bmatrix}, \quad B = \begin{bmatrix} -2{\cdot}3964 \\ 0 \\ -4{\cdot}2108 \end{bmatrix}$$

The open-loop poles are $-3{\cdot}2638$, $0{\cdot}8550 \pm 2{\cdot}4978i$.

The step response of this system is represented in Fig. 3. The system is unstable and oscillatory with a period about 2 s.

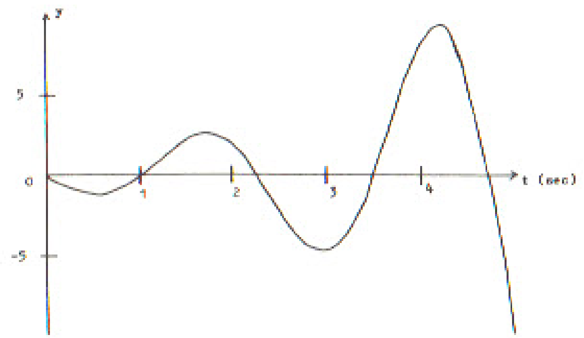

Figure 3. Open-loop step response of the turbo-generator.

Influence of the sampling period T

Let us consider a continuous control $u_c = K_c x_c$, where K_c is given by (32), with

$$\alpha = 0, \quad R_c = 64, \quad Q_c = 1, \quad \gamma = 1$$

The Nichols chart of the transfer function $-K_c(sI - A)^{-1}B$ is represented in Fig. 4 (case $T = 0$). The phase margin measured on this plot is $\Phi_R = 61°$, i.e. for this application, Φ_R is barely bigger than the phase margin $\Phi = 60°$ guaranteed by the theory.

Now the system is sampled at the period T. The Euler transform (37) is used, so that one obtains the discrete-time model (7) with F and G verifying (29) and (30). The discrete-time control (9) is used, where K is given by (15)–(17) with

$$\rho = 1, \quad R = 64, \quad Q = 1, \quad \gamma = 1$$

The Nichols charts of the transfer functions $-K(zI - F)^{-1}G$ are represented in Fig. 4 for various values of T (of course, these Nichols charts have been obtained via the classical w-transform). One notices that the robustness decreases as T increases. One can also note that the phase margin can be increased by multiplying the control by a well-chosen gain $1/\gamma$. In this manner, an optimum phase margin Φ_R^* is obtained, for a corresponding gain $1/\gamma_R^*$. The values Φ_R, Φ_R^* and γ_R^* have to be compared with the values Φ, Φ^* and γ^* of the above theory. This comparison is in the Table.

The values Φ, Φ^* and γ^* of the theory are therefore good approximations of the actual values Φ_R, Φ_R^* and γ_R^*.

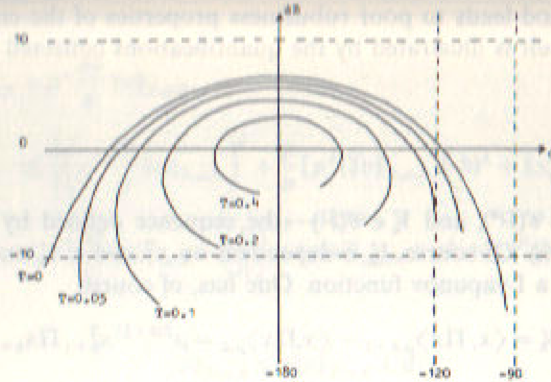

Figure 4. Nichols charts of the controlled system, for T varying 0–0·4 s.

T (s)	Φ(°)	Φ_R(°)	Φ^*(°)	Φ_R^*(°)	γ^*	γ_R^*
0·00	60·0	61·0	90·0	90·0	0·00	0·00
0·05	55·5	56·2	68·6	69·0	0·36	0·35
0·10	50·1	50·6	57·8	58·1	0·53	0·56
0·20	39·0	39·4	41·8	42·2	0·74	0·74
0·40	22·3	22·5	22·7	23·4	0·92	0·92

Comparison of Φ, Φ^* and γ^* with Φ_R, Φ_R^* and γ_R^*.

Now suppose that a given phase margin of, say $\Phi_R \geq 50°$ is wanted. By application of formula (40) with $\Delta\Phi = \pi/18$, one obtains $T^* = 0{\cdot}13$ s. In the table one notices that the desired phase margin is obtained for $T < 0{\cdot}1$ s. Therefore, T^* is a good indication for choosing T (we suggest to choose $T < T^*/2$ for a good safety).

Note that the performance of the closed-loop system is improved (i.e. it becomes faster) with $\gamma^* \leq \gamma < 1$, in comparison with the case $\gamma = 1$. The robustness, in the sense considered in the paper (i.e. gain margin and phase margin) is also improved. This seems very surprising, but as the frequency at 0 db becomes greater, the robustness in the face of high-frequency model uncertainties (such that unmodelled delays) is in fact reduced. In the continuous-time case, for instance, the closed-loop system becomes infinitely fast when $\gamma \to 0$; the gain margin becomes $]0, \infty[$ and the phase margin becomes 90°; but the 'delay margin' is reduced to 0.

6. Concluding remarks

The general Theorems 1, 2 and 3 of ρ-stability have led to a set of conditions allowing direct quantification of the robustness properties of discrete-time LQ regulators (Theorems 4 and 5). Of course, they can be applied to other purposes; Bourlès (1987 b) has used the continuous-time version of Theorem 2 to obtain a powerful stability theorem for a delay system.

The robustness results of this paper offer a generalization of results which have appeared earlier in the literature. Equation (40) is especially recommended for choosing the sampling period (we suggest a choice of $T \leq T^*/2$). This formula shows clearly that even if an exact discretization has been done—see (28)—the choice of a too large sampling period leads to poor robustness properties of the controlled system. This theoretical result is illustrated by the quantifications collected in the Table.

Appendix A

Proof of Theorem 1

Let $x^0 \in \mathbb{R}^n$, $v \in \mathscr{C}(\mathbb{R}^n)$, and $V_k \in \mathscr{C}(\mathbb{R})$—the sequence defined by $V_k = \langle x, \Pi x\rangle_{\rho,k}$, where $x_k \triangleq \varphi(x^0, v, k)$. Of course, V_k is dependent on x^0 and v; V_k may be interpreted in the following as a Lyapunov function. One has, of course

$$V_{k+1} - V_k = \langle x, \Pi x\rangle_{\rho,k+1} - \langle x, \Pi x\rangle_{\rho,k} = \rho^{2(k+1)} x_{k+1}^{\mathrm{T}} \Pi x_{k+1} \geqslant 0$$

One obtains

$$V_{k+1} - V_k = x^{0\mathrm{T}}\Pi x^0 + \sum_{i=0}^{k} \rho^{2i}[\rho^2 x_{i+1}^{\mathrm{T}}\Pi x_{i+1} - x_i^{\mathrm{T}}\Pi x_i]$$

so that by (3)

$$V_{k+1} - V_k = x^{0\mathrm{T}}\Pi x^0 + \rho^2\langle \mathscr{F}(x), \Pi\mathscr{F}(x)\rangle_{\rho,k} - \langle x, \Pi x\rangle_{\rho,k} + 2\rho^2\langle \mathscr{F}(x), \Pi v\rangle_{\rho,k} + \rho^2\langle v, \Pi v\rangle_{\rho,k}$$

and by (4)

$$V_{k+1} - V_k \leqslant x^{0\mathrm{T}}\Pi x^0 - \langle x, \Sigma x\rangle_{\rho,k} + 2\rho^2\langle \mathscr{F}(x), \Pi v\rangle_{\rho,k} - \rho^2\langle v, \Pi v\rangle_{\rho,k}$$

As $V_{k+1} - V_k \geqslant 0$, one has

$$\langle x, \Sigma x\rangle_{\rho,k} \leqslant x^{0\mathrm{T}}\Pi x^0 + 2\rho^2\langle \mathscr{F}(x), \Pi v\rangle_{\rho,k} - \rho^2\langle v, \Pi v\rangle_{\rho,k} \tag{A 1}$$

First, let us consider the case $v = 0$; (A 1) becomes

$$\langle x, \Sigma x\rangle_{\rho,k} \leqslant x^{0\mathrm{T}}\Pi x^0 \tag{A 2}$$

By (A 2), $\Sigma^{1/2}x \in l_\rho^2(\mathbb{R}^n)$. Therefore, $x \in l_\rho^2(\mathbb{R}^n)$ if $(\Sigma^{1/2}, \mathscr{F})$ is ρ-detectable, i.e.

$$\sum_{k=0}^{\infty} \|\rho^k x_k\|^2 < \infty \tag{A 3}$$

By (A 3), $(\rho^k x_k)$ converges to zero, so that 0 is globally ρ-attractive for $\mathscr{S}$.

By (A 2), $\|\Sigma^{1/2}x\|_{2,\rho} \leqslant (\|\Pi\|)^{1/2}\|x^0\|$. Let $\varepsilon > 0$; if $(\Sigma^{1/2}, \mathscr{S})$ is strongly ρ-detectable, there exists $\eta = 0$ such that $\|x\|_{2,\rho} \leqslant \varepsilon$ if $\|\Sigma^{1/2}x\|_{2,\rho} \leqslant \eta$, i.e. if $\|x^0\| \leqslant (1/\|\Pi\|^{1/2})\ \eta \triangleq \eta'$ (for $\Pi = 0$). As $\forall k$, $\|\rho^k x_k\| \leqslant \|x\|_{2,\rho}$, one obtains $\|\rho^k x_k\| \leqslant \varepsilon$, for $\|x^0\| \leqslant \eta'$. Therefore, $\mathscr{S}$ is LρS, and finally GAρS.

Now, let us consider the general case $(v \neq 0)$ with $\Sigma > 0$ and $\gamma_\rho(\mathscr{F}) < \infty$, so that there exist constants $a > 0$ and $b \geqslant 0$ such that $\forall x \in \mathscr{C}(\mathbb{R}^n)$ and $\forall k \geqslant 0$

$$\|\mathscr{F}(x)\|_{2,\rho,k} \leqslant a\|x\|_{2,\rho,k} + b$$

By (A 1), one obtains with $\sigma \triangleq \underline{\sigma}(\Sigma)$ and $p \triangleq \bar{\sigma}(P)$

$$\|x\|_{2,\rho,k}^2 - 2\rho^2\frac{ap}{\sigma}\|x\|_{2,\rho,k}\|v\|_{2,\rho,k} \leqslant \frac{p}{\sigma}[\|x^0\|^2 + 2\rho^2 b\|v\|_{2,\rho,k} + \rho^2\|v\|_{2,\rho,k}^2]$$

Therefore

$$\|x\|_{2,\rho,k} - \rho^2\frac{ap}{\sigma}\|v\|_{2,\rho,k} \leqslant \left\{\left(\rho^2\frac{ap}{\sigma}\|v\|_{2,\rho,k}\right)^2 + \frac{p}{\sigma}[\rho^2(\|v\|_{2,\rho,k} + b)^2 + \|x^0\|^2]\right\}^{1/2} \leqslant \rho^2\frac{ap}{\sigma}\|v\|_{2,\rho,k} + \left(\frac{p}{\sigma}\right)^{1/2}(\rho\|v\|_{2,\rho,k} + \rho b + \|x^0\|)$$

so that finally

$$\|x\|_{2,\rho,k} \leqslant c\|v\|_{2,\rho,k} + d$$

where

$$c \triangleq 2\rho^2\frac{ap}{\sigma} + \rho\left(\frac{p}{\sigma}\right)^{1/2} \quad \text{and} \quad d \triangleq \left(\frac{p}{\sigma}\right)^{1/2}(\rho b + \|x^0\|)$$

□

Appendix B

Proof of Theorem 2

Let ε be such that

$$0 < \varepsilon < \underline{\sigma}(\Sigma) - 2\rho\,\gamma^0(\mathscr{F}_2)\bar{\sigma}(\Pi)$$

so that

$$0 < \varepsilon I < \Sigma \leqslant \Pi$$

and moreover

$$\gamma_\rho^0(\mathscr{F}_2) < \frac{\underline{\sigma}(\Sigma) - \varepsilon}{2\rho\bar{\sigma}(\Pi)} = \frac{\underline{\sigma}(\Sigma - \varepsilon I)}{2\rho\bar{\sigma}(\Pi)} \leqslant \frac{\underline{\sigma}(\Sigma - \varepsilon I)}{2\rho(\bar{\sigma}(\Pi)\bar{\sigma}(\Pi - \varepsilon I))^{1/2}}$$

One obtains

$$(\langle \mathscr{F}_2(x), \Pi\mathscr{F}_2(x)\rangle_{\rho,k})^{1/2} \leqslant \gamma_\rho^0(\mathscr{F}_2)(\bar{\sigma}(\Pi))^{1/2}\|x\|_{2,\rho,k} \leqslant \frac{1}{2\rho}\frac{\underline{\sigma}(\Sigma - \varepsilon I)}{(\bar{\sigma}(\Pi - \varepsilon I))^{1/2}}\|x\|_{2,\rho,k}$$

One has also

$$\|x\|_{2,\rho,k}^2 \leqslant \frac{1}{\underline{\sigma}(\Sigma - \varepsilon I)}\langle x, (\Sigma - \varepsilon I)x\rangle_{\rho,k}$$

and

$$(\bar{\sigma}(\Pi - \varepsilon I))^{1/2}\|x\|_{2,\rho,k} \geqslant (\langle x, (\Pi - \varepsilon I)x\rangle_{\rho,k})^{1/2}$$

so that

$$(\langle \mathscr{F}_2(x), \Pi\mathscr{F}_2(x)\rangle_{\rho,k})^{1/2} \leqslant \frac{1}{2\rho}\frac{\langle x, (\Sigma - \varepsilon I)x\rangle_{\rho,k}}{(\langle x, (\Pi - \varepsilon I)x\rangle_{\rho,k})^{1/2}} \tag{B 1}$$

By the triangular inequality

$$(\langle \mathscr{F}(x), \Pi\mathscr{F}(x)\rangle_{\rho,k})^{1/2} \leqslant (\langle \mathscr{F}_1(x), \Pi\mathscr{F}_1(x)\rangle_{\rho,k})^{1/2} + (\langle \mathscr{F}_2(x), \Pi\mathscr{F}_2(x)\rangle_{\rho,k})^{1/2}$$

so that

$$\rho(\langle \mathscr{F}(x), \Pi\mathscr{F}(x)\rangle_{\rho,k})^{1/2} \leqslant (\langle x, (\Pi-\Sigma)x\rangle_{\rho,k})^{1/2} + \frac{1}{2}\frac{\langle x, (\Sigma-\varepsilon I)x\rangle_{\rho,k}}{(\langle x, (\Pi-\varepsilon I)x\rangle_{\rho,k})^{1/2}}$$

Now, apply the inequality (holding for $a>0, b\geqslant 0$)

$$(a-b)^{1/2} + \frac{b}{2\sqrt{a}} \leqslant \sqrt{a}$$

with

$$a = \langle x, (\Pi-\varepsilon I)x\rangle_{\rho,k}$$

$$b = \langle x, (\Sigma-\varepsilon I)x\rangle_{\rho,k}$$

One obtains finally

$$\rho(\langle \mathscr{F}(x), \Pi\mathscr{F}(x)\rangle_{\rho,k})^{1/2} \leqslant (\langle x, (\Pi-\varepsilon I)x\rangle_{\rho,k})^{1/2}$$

so that (4) is satisfied (with Σ replaced by εI). □

Appendix C

Proof of Proposition 2

One has

$$\begin{aligned}\langle \mathscr{F}_1(x), \Pi\mathscr{F}_1(x)\rangle_{\rho,k} &= \|\Pi^{1/2}\mathscr{P}_k\mathscr{F}_1(x)\|^2_{2,\rho}\\ &= \|\Pi^{1/2}\mathscr{P}_k\mathscr{F}_1(\mathscr{P}_k x)\|^2_{2,\rho} \quad \text{by causality}\\ &\leqslant \|\Pi^{1/2}\mathscr{F}_1(\mathscr{P}_k x)\|^2_{2,\rho}\end{aligned}$$

Moreover, one has

$$\Pi^{1/2}\mathscr{F}_1(\mathscr{F}_k x) = T * \mathscr{P}_k x$$

where T is the sequence of $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ defined by $T_k = \Pi^{1/2}H_k$. Therefore, one obtains

$$\langle \mathscr{F}_1(x), \Pi\mathscr{F}_1(x)\rangle_{\rho,k} \leqslant \|T*(\mathscr{P}_k x)\|^2_{2,\rho}$$

Let $\hat{X}_k$ be the z-transform of $\mathscr{P}_k x$. By the Parseval equality, one obtains

$$\begin{aligned}\|T*(\mathscr{P}_k x)\|^2_{2,\rho} &= \frac{1}{2\pi i}\int_{C_{1/\rho}} \|\hat{T}(z)\hat{X}_k(z)\|^2 z^{-1}\,dz\\ &= \frac{1}{2\pi i}\int_{C_{1/\rho}} \hat{X}_k^*(z)\hat{H}^*(z)\Pi\hat{H}(z)\hat{X}_k(z)z^{-1}\,dz\end{aligned}$$

so that by (5)

$$\begin{aligned}\|T*(\mathscr{P}_k x)\|^2_{2,\rho} &\leqslant \frac{1}{2\pi i}\int_{C_{1/\rho}} \hat{X}_k^*(z)U\hat{X}_k(z)z^{-1}\,dz\\ &= \langle x, Ux\rangle_{\rho,k}\end{aligned}$$

□

Appendix D

Proof of Theorem 5

For $\hat{M}(z) = I_n$, the condition (13) becomes for $\Pi = P$ and $\Sigma = Q$

$$\rho^2\hat{H}^*(z)P\hat{H}(z) \leqslant P - Q \qquad \text{(D 1)}$$

with $\hat{H}(z) \triangleq F + G\hat{L}(z)K$

We will show in the following that (D 1) is equivalent to (18).

One has by (15 and 17)

$$\hat{H}^*(z)P\hat{H}(z) = F^{\mathrm{T}}PF - \frac{\rho^2}{\gamma}C^{\mathrm{T}}D(z)C$$

with

$$C \triangleq (R + \rho^2 G^{\mathrm{T}}PG)^{-1/2}G^{\mathrm{T}}PF$$

$$D(z) \triangleq S^{1/2}\hat{L}^*(z)S^{-1/2} + S^{-1/2}\hat{L}(z)S^{1/2} - \frac{\rho^2}{\gamma}S^{1/2}\hat{L}^*(z)G^{\mathrm{T}}PG\hat{L}(z)S^{1/2}$$

where $S \triangleq (R + \rho^2 G^{\mathrm{T}}PG)^{-1}$.

By (16), one obtains

$$\begin{aligned}\rho^2\hat{H}^*(z)P\hat{H}(z) &= P + \rho^4 C^{\mathrm{T}}C - Q - \frac{\rho^4}{\gamma}C^{\mathrm{T}}D(z)C\\ &= P - Q + \rho^4 C^{\mathrm{T}}\left[I_m - \frac{1}{\gamma}E(z)\right]C\end{aligned}$$

with

$$E(z) \triangleq S^{1/2}\hat{L}^*(z)S^{-1/2} + S^{-1/2}\hat{L}(z)S^{1/2} + \frac{\rho^2}{\gamma}S^{-1/2}\hat{L}^*(z)G^{\mathrm{T}}PG\hat{L}(z)S^{1/2}$$

Therefore, (D 1) is equivalent to $E(z) \leqslant \gamma I_m$, i.e. to

$$\hat{L}^*(z)R + R\hat{L}(z) - \gamma R \geqslant \rho^2\left[\frac{1}{\gamma}\hat{L}^*(z)G^{\mathrm{T}}PG\hat{L}(z) - \gamma G^{\mathrm{T}}PG - G^{\mathrm{T}}PG\hat{L}(z) - \hat{L}^*(z)G^{\mathrm{T}}PG\right] \qquad \text{(D 2)}$$

One has

$$\hat{L}^*(z)R + R\hat{L}(z) - \gamma R = -\gamma\left[I_m - \frac{1}{\gamma}\hat{L}^*(z)\right]R\left[I_m - \frac{1}{\gamma}\hat{L}(z)\right] + \frac{1}{\gamma}\hat{L}^*(z)R\hat{L}(z)$$

and the second member of (D 2) is equal to

$$\rho^2\gamma\left[I_m - \frac{1}{\gamma}\hat{L}^*(z)\right]G^{\mathrm{T}}PG\left[I_m - \frac{1}{\gamma}\hat{L}(z)\right]$$

so that (D 2) becomes

$$\hat{L}^*(z)R\hat{L}(z) \geqslant \gamma^2\left[I_m - \frac{1}{\gamma}\hat{L}(z)\right]^*(R + \rho^2 G^{\mathrm{T}}PG)\left[I_m - \frac{1}{\gamma}\hat{L}(z)\right]$$

This condition is equivalent to (18) if $\hat{L}(z)$ is invertible. In the case where $R = \mathrm{diag}\,(rj)$

and $L(z) = \mathrm{diag}\,(\hat{l}_j(z))$, (18) is satisfied if

$$\left|1 - \frac{\gamma}{l_j(z)}\right| \leqslant a_j \tag{D 3}$$

for all $z \in C_{1/\rho}$ and $j \in \{1, ..., m\}$. This inequality (D 3) is equivalent to

$$|\hat{l}_j(z) - \gamma|^2 \leqslant a_j^2 |\hat{l}_j(z)|^2$$

i.e. to

$$\left|l_j(z) - \frac{\gamma}{1 - a_j^2}\right| \leqslant \frac{\gamma aj}{1 - a_j^2}$$

Note that as $a_j < 1$, this inequality cannot be verified if $\hat{l}_j(z) = 0$, i.e. if $\hat{L}(z)$ is not invertible. □

References

Anderson, B. D. O., and Moore, J. B., 1971, *Linear Optimal Control* (New York: Prentice-Hall).

Bénéjean, R., 1986, Modèles mathématiques d'un groupe turbo-alternateur connecté à un réseau de transport électrique. Bulletin de la DER, EDF, Série C, No. 4, 63–107.

Bourlès, H., 1982, Qualification et amélioration de la robustness des régulateurs multivariables, avec application au pilotage d'un avion. Thèse de Docteur-Ingénieur, INPG-ONERA, Grenoble, N.T. ONERA No. 1983–2; 1984, Stabilité des systèmes linéaires perturbés: Robustesse. *Rev. Autom. Inf. Res. Oper*, **18,** 297–314; 1986 a, Stabilité de degré α des systèmes régis par une équation differentielle fonctionnelle. *APII, AFCET,* **19,** 455–473; 1986 b, Stabilité et robustesse de degré α; application à la mise au point d'un régulateur. *Proc. Colloque SMAI 'L'Automatique pour l'Aéronautique'*, Paris, Cepadues-Editions; 1987 a, α-Stability and robustness of large-scale interconnected systems. *International Journal of Control*, **45,** 2221–2232; 1987 b, α-Stability of systems governed by a functional differential equation—extension of results concerning linear delay systems, *Ibid.*, **45,** 2233–2234.

Bourlès, H., and Mercier, O. L., 1982, Marges de stabilité et robustesse structurelle généralisées des régulateurs linéaires quadratiques multivariables. *Rev. Autom. Inf. Res. Oper.*, **16,** 49–70.

Desoer, C. A., and Vidyasagar, M., 1975, *Feedback Systems: Input–Output Properties* (New York: (Academic Press).

Dorato, P., (editor), 1987, *Robust Control* (New York: I.E.E.E. Press).

Irving, E., Khaletsky, D., Bendotti, P., Hamada, A., Raluy, G., 1988, Optimal adaptive control insensitive to complexity for terminal voltage and speed controls of electric turbo-generator units. *Symposium on theory and applications of adaptive control*, Lodge Kananaski, Alberta.

Joannic, Y., 1983, La Robustesse des systèmes dynamiques à temps discret. Application au pilotage numérique multivariable des avions d'armes. *IDN-ONERA*, Lille, N.T. ONERA No. 1984–2, ESA-TT-906.

Joannic, Y., and Mercier, O. L., 1983, Quantification de la robustesse des systèmes non linéaires à commande échantillonnée et à perturbations dynamiques, *Recherche Aérospacial*, **6,** 279–394; 1986, Théorie et mise au point de régulateurs multivariables discrete assurant une poursuite robuste. *Proc. Colloque SMAI 'L'Automatique pour l'Aéronautique'*, Cepadues Editions.

Kwakernak, H., and Sivan, R., 1972, *Linear Optimal Control Systems* (New York: Wiley).

Mercier, O. L., Joannic, Y., and Moreigne, O., 1983, Mise au point interactive des régulateurs linéaires quadratiques par action directe et sélective sur les pôles du système commandé, *Proc. 1er Symposium 'Informatique Appliquée de l'IASTED*, Lille, Vol. 2, pp. 201–206.

Safonov, M. G., 1980, *Stability and Robustness of Multivariable Feedback Systems* (Cambridge, Mass: MIT Press).

Safonov, M. G., and Athans, M., 1977, Gain and phase margin for multiloop LQG regulators. *I.E.E.E. Transactions on Automatic Control*, **22,** 173–179.

Vidyasagar, M., 1981, *Input–Output Analysis of Large-Scale Interconnected Systems* (Berlin: Springer-Verlag).

Willems, J. C., 1971, *The Analysis of Feedback Systems* (Cambridge, Mass: MIT Press).

Zames, G., 1966, On the input-output stability of time-varying nonlinear feedback systems. Part II: Conditions involving circles in the frequency plane and sector nonlinearities. *I.E.E.E. Transactions on Automatic Control*, **11,** 465–476.