Gain and phase margins of the guaranteed cost regulator

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Closed-loop stability conditions are given for systems with both structured and unstructured uncertainties. In particular, it is shown that the guaranteed cost regulator taking into account the structured uncertainties, can tolerate unstructured uncertainties as well. If the systems dynamics matrix only is uncertain, this controller provides the same stability margins as the standard LQ optimal regulator, for all admissible parameter variations. If, in addition, uncertainties enter into the input matrix, stability margins depend on the uncertainties on B. Furthermore, these results are extended to the a-stability case.

1. Introduction

Robustness properties of the optimal linear quadratic regulator, expressed in terms of the well-known LQ stability margins, have been studied in several works (see e.g. Anderson and Moore 1971, Safonov and Athans 1977, Lehtomaki *et al.* 1981). Furthermore, it has been shown by Bourlès (1986) that these margins remain valid even if an unstructured multiplicative uncertainty is introduced in the standard LQ scheme, provided a sufficient norm condition on the uncertainty is satisfied.

In recent literature the standard LQ context has been proved to be the natural framework for designing robust control systems (i.e. preserving desirable closed-loop specifications) in the presence of structured uncertainties expressed in terms of parameter variations (see e.g. Petersen and Hollot 1986, Bernstein and Haddad 1988, Kosmidou 1990 and related references).

In particular, the guaranteed cost control approaches (Chang and Peng 1972, Vinkler and Wood 1979, Kosmidou and Bertrand 1987, Bernstein and Haddad 1988, Kosmidou 1990, Kosmidou *et al.* 1991) require the minimization of a quadratic cost functional whose value is guaranteed to be bounded above in presence of system parameter variations of a given class. Thus, the closed-loop system's performance degradation due to the uncertainties is guaranteed to be less than this upper bound. Another advantage of guaranteed cost control methods is the fact that the controllers obtained are linear and with constant gains resulting from the solutions of modified Riccati equations, i.e. Riccati equations containing additional terms which depend on the uncertainties.

Some questions arising about guaranteed cost controllers are as follows.

(*a*) Do they have the same or equivalent stability margins as the standard LQ regulators?

(*b*) What will happen if an unstructured multiplicative uncertainty is introduced into the standard guaranteed cost control scheme?

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These questions are studied in the present paper.

In § 2 the guaranteed cost control approach is described. The robust stability condition in the presence of unstructured uncertainty is given in § 3. The guaranteed cost control stability margins are studied in § 4 and extended to the a-stability case in § 5; § 6 deals with discussion and concluding remarks.

2. The guaranteed cost control problem

Consider the linear uncertain system described by the state equation

$$\dot{x}(t) = (A_0 + \Delta A)x(t) + (B_0 + \Delta B)u(t), \quad t \in [0, \infty) \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector and $A_0 \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{n\times m}$ are the nominal system matrices. Structured uncertainties taking into account plant parameter variations are described as follows:

$$\Delta A = \sum_{i=1}^{k} A_i r_i(t) \tag{2.2}$$

$$\Delta B = \sum_{i=1}^{l} B_i q_i(t) \tag{2.3}$$

where A_i, $i = 1, 2, \ldots, k$ and B_i, $i = 1, 2, \ldots, l$ are constant matrices of appropriate dimensions and $r_i(t)$, $q_i(t)$ are uncertain parameters, possibly time-varying (provided time variation is slow with respect to system dynamics). The only information about uncertainties is that they are Lebesgue measurable vector functions belonging to known and bounded compact sets **R** and **Q** where

$$\mathbf{R} = \{r \in \mathbb{R}^k, |r_i| \leqslant \bar{r}, i = 1, 2, \ldots, k\}; \quad \bar{r} \geqslant 0 \tag{2.4}$$

$$\mathbf{Q} = \{q \in \mathbb{R}^l, |q_i| \leqslant \bar{q}, i = 1, 2, \ldots, l\}; \quad \bar{q} \geqslant 0 \tag{2.5}$$

Furthermore, uncertainties are assumed to be of the 'rank-l' type and thus

$$A_i = d_i e_i^{\mathrm{T}}, \quad i = 1, 2, \ldots, k \tag{2.6}$$

$$B_i = f_i g_i^{\mathrm{T}}, \quad i = 1, 2, \ldots, l \tag{2.7}$$

with d_i, e_i, $f_i \in \mathbb{R}^n$ and $g_i \in \mathbb{R}^m$. It should be noted that the above decomposition is not unique.

The following symmetric positive semi-definite matrices are defined:

$$T = \bar{r}\sum_{i=1}^{k} d_i d_i^{\mathrm{T}} \quad U = \bar{r}\sum_{i=1}^{k} e_i e_i^{\mathrm{T}}$$
$$W = \bar{q}\sum_{i=1}^{l} f_i f_i^{\mathrm{T}} \quad V = \bar{q}\sum_{i=1}^{l} g_i g_i^{\mathrm{T}} \tag{2.8}$$

The system's performance is described by the quadratic cost functional

$$J(x(t), u(t), t) = \int_0^{\infty} [x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)]\,dt \tag{2.9}$$

where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are positive definite weighting matrices.

Moreover it is assumed that (A_0, B_0) form a controllable pair, $(A_0, Q^{1/2})$ form an observable pair and $x(t)$ is available for feedback.

For the above system the design objective is to find a *guaranteed cost control* law of the form

$$\tilde{u}(t) = Kx(t) \tag{2.10}$$

such that the corresponding value of the performance index (2.9) has an upper bound of the form

$$J(x(t), \tilde{u}(t), t) \leqslant \tilde{V} \tag{2.11}$$

for all $r(t)$, $q(t)$ consistent with (2.4), (2.5).

It has been shown in (Kosmidou *et al*. 1991) that such a control law is of the form

$$\tilde{u}(t) = -R^{-1}B_0^{\mathrm{T}}Px(t) \tag{2.12}$$

where P is the symmetric positive definite solution, if it exists, of the modified Riccati equation

$$PA_0 + A_0^{\mathrm{T}}P - P[B_0R^{-1}B_0^{\mathrm{T}} - B_0R^{-1}VR^{-1}B_0^{\mathrm{T}} - W - T]P + U + Q = 0 \tag{2.13}$$

Alternative forms of the modified Riccati equation, depending on different assumptions on the uncertainties, have been obtained by Chang and Peng (1972), Vinkler and Wood (1979), Kosmidou and Bertrand (1987) and Kosmidou (1990).

3. Stability robustness in presence of unstructured uncertainty

The results of this Section have been given by Bourlès (1984, 1987).

Consider a state-space system description in which the uncertainty is described by a multiplicative term at the system's input. Such an unstructured uncertainty may take into account modelling errors, such as neglected dynamics, non-linearities, parasitic delays, etc. In other words, for the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \in [0, \infty) \tag{3.1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, the control law is of the form

$$u(t) = (L^*Kx)(t) \tag{3.2}$$

where $L(t) \in \mathbb{R}^{m\times m}$ denotes the unstructured uncertainty impulse response matrix, $K \in \mathbb{R}^{m\times n}$ is the constant gain matrix and (*) denotes the convolution product. (By some abuse of language, the convolution product $(L^*y)(t)$ will be denoted $L(t)^*y(t)$, even if L is a distribution). It is assumed that the elements of matrix L are bounded Radon measures.

Using the control law (3.2) the closed-loop system dynamics are

$$\dot{x}(t) = [F(x)](t) \tag{3.3}$$

where

$$[F(x)](t) := [A + BL^*K]x(t) \tag{3.4}$$

Equation (3.3) is a functional differential equation (Hale 1977).

A stability condition for system (3.3) is given by means of the following Theorem.

Theorem 3.1: *The closed-loop system (3.3) is asymptotically stable if there exist positive definite matrices $\tilde{P} \in \mathbb{R}^{n\times n}$ and $\tilde{Q} \in \mathbb{R}^{n\times n}$ such that*

$$\langle x, \tilde{P}F(x) + \tilde{Q}/2\rangle_t \leq 0 \tag{3.5}$$

$\forall\, x \in \mathbb{R}^n,\ t \in [0, \infty)$.

Sketch of the proof: In (3.5), $\langle \cdot, \cdot \rangle_t$ denotes the 'truncated inner product' in the extended space $\mathbb{L}_{2e}$ (Desoer and Vidyasagar 1975), i.e.

$$\langle h, y\rangle_t := \int_0^t h(\tau)^{\mathrm{T}} y(\tau)\,\mathrm{d}\tau$$

Consider the Lyapunov function

$$V_L(x) = x^{\mathrm{T}}(t)\tilde{P}x(t)$$

Then,

$$\dot{V}_L(x) = 2x^{\mathrm{T}}\tilde{P}F(x)$$

By (3.5) one has

$$\int_0^t \dot{V}_L(x(\tau))\,\mathrm{d}\tau \leq -\int_0^t x^{\mathrm{T}}(\tau)\tilde{Q}x(\tau)\,\mathrm{d}\tau$$

i.e.

$$V_L(x(t)) - V_L(x(0)) \leq -\int_0^t x^{\mathrm{T}}(\tau)\tilde{Q}x(\tau)\,\mathrm{d}\tau$$

Therefore, $x \in \mathbb{L}_2$. Now, it is easily proved, using the above assumption, that $\dot{x} \in \mathbb{L}_2$. As a result, $x(t) \to 0$, as $t \to \infty$.

This Theorem is proven in more detail in Bourlès (1984, 1987).

For system (3.3) we can write

$$[F(x)](t) = [T^* x](t) \tag{3.6}$$

where

$$T(t) := A\delta(t) + BL(t)K \tag{3.7}$$

and $\delta(t)$ is the Dirac impulse. Let us denote $\hat{T}(s)$, $\hat{L}(s)$ the Laplace transforms of $T(t)$ and $L(t)$, respectively. Then (3.7) yields

$$\hat{T}(s) = A + B\hat{L}(s)K \tag{3.8}$$

and the sufficient condition of Theorem 3.1 obtains the form

$$\hat{T}^*(s)\tilde{P} + \tilde{P}\hat{T}(s) \leq -\tilde{Q} \tag{3.9}$$

$\forall s \in I_0 = \{s \colon s = j\omega\}$, where $(\cdot)^*$ denotes the conjugate transpose of $(\cdot)$.

4. Stability robustness of the guaranteed cost control

The sufficient condition of the previous section will now be applied in order to investigate stability robustness of the guaranteed cost controller with respect to unstructured multiplicative uncertainty. In other words, we will look for conditions under which the closed-loop system obtained by using the guaranteed cost control method of § 2 remains stable in presence of a multiplicative uncertainty $\hat{L}(s)$.

The system under consideration is

$$\dot{x}(t) = (A_0 + \Delta A)x(t) + (B_0 + \Delta B)u(t) \tag{2.1$'$}$$

in which the control law

$$u(t) = (L^* Kx)(t) \tag{3.2$'$}$$

is applied, where L denotes an unstructured uncertainty, as described in § 3, K is the guaranteed cost controller gain,

$$K = -R^{-1}B_0^{\mathrm{T}}P \tag{2.12$'$}$$

and P satisfies the guaranteed cost equation (2.13).

According to Theorem 3.1, a sufficient condition for the closed-loop system to be asymptotically stable, is

$$[(A_0 + \Delta A) + (B_0 + \Delta B)\hat{L}(s)K]^* P + P[(A + \Delta A) + (B_0 + \Delta B)\hat{L}(s)K] \leq -Q \tag{4.1}$$

where P and Q satisfy (2.13) and where ΔA, ΔB are given by (2.2), (2.3). The following Lemma will be used in the sequel.

Lemma 4.1: *Let*

$$\bar{A}(P) := PTP + U \tag{4.2}$$

and

$$\bar{B}(P) := PWP + PB_0R^{-1}\hat{L}^*(s)V\hat{L}(s)R^{-1}B_0^{\mathrm{T}}P \tag{4.3}$$

with T, U, W, V given by (2.8). The following inequalities can be demonstrated

$$2x^{\mathrm{T}}(t)P\Delta Ax(t) \leq x^{\mathrm{T}}(t)\bar{A}(P)x(t) \tag{4.4}$$

$$-x^*(s)P\Delta B\hat{L}(s)R^{-1}B_0^{\mathrm{T}}Px(s) - x^*(s)PB_0R^{-1}\hat{L}^*(s)\Delta B^{\mathrm{T}}Px(s) \leq x^*(s)\bar{B}(P)x(s) \tag{4.5}$$

$\forall x(t), x(s) \in \mathbb{R}^n$, $r \in \mathbf{R}$, $q \in \mathbf{Q}$.

Proof: The proof is given in the Appendix. □

The following Theorem can now be demonstrated.

Theorem 4.1: *The closed-loop system (2.1)$'$, (3.2)$'$ remains stable in presence of unstructured uncertainty $\hat{L}(s)$ if the following condition is satisfied*

$$2\inf_{s\in I_0}\mu[H(s)] \geq 1 + \bar{\sigma}(X)\{(\|J(s)\|_\infty + 1)^2 - 1\} \tag{4.6}$$

where

$$H(s) := R^{\frac{1}{2}}\hat{L}(s)R^{-\frac{1}{2}} \tag{4.7}$$

$$X(s) := R^{-\frac{1}{2}}VR^{-\frac{1}{2}} \tag{4.8}$$

$$J(s) := H(s) - I \tag{4.9}$$

$\|\cdot\|_\infty$ denotes the infinity norm of matrix $(\cdot)$, $\bar{\sigma}(\cdot)$ denotes the maximum singular value of $(\cdot)$, $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of $(\cdot)$ and $\mu(\cdot) := (\frac{1}{2})\lambda_{\min}[(\cdot) + (\cdot)^*]$.

Proof: The proof is given in the Appendix. □

Remark 4.1: In the absence of multiplicative uncertainty, i.e. for $\hat{L}(s) = I$, $H(s) = I$, $J(s) = 0$, condition (4.6) reduces to $2 > 1$ which is obviously always true. In other words, the guaranteed cost controller stabilizes the closed-loop system for all r_i, q_i consistent with (2.4), (2.5). □

In order to give an easier interpretation of the stability condition (4.6), let us consider the following particular cases.

4.1. *Gain variation*

Suppose that the unstructured uncertainty corresponds to a constant gain variation g, i.e. $\hat{L}(s) = \hat{L} = gI$, and the control weighting matrix R is chosen to be diagonal such that $R = \rho I$, $\rho > 0$. Hence we have $H(s) = H = gI$, $X = V/\rho$ and $J(s) = J = (g-1)I$. Define

$$a := \bar{\sigma}(V)/\rho \tag{4.10}$$

Condition (4.6) obtains the form

$$2g \geqslant 1 + a\{(|g-1| + 1)^2 - 1\} \tag{4.11}$$

and finally one obtains the following Corollary.

Corollary 4.1—Gain margin: *The gain margin of the closed-loop system obtained by the guaranteed cost controller is given by the expression*

$$GM = \{(1/a)[2a + 1 - (1 + 3a + a^2)^{\frac{1}{2}}],\quad (1/a)[1 + (1 - a + a^2)^{\frac{1}{2}}]\} \tag{4.12}$$

Proof: The proof is given in the Appendix. □

It is obvious from (A 17), (A 19) that unity belongs to the above interval. Besides, the value of $a = \bar{\sigma}(V)/\rho$ is of critical importance in (4.12). Gain margin as a function of a is given in Figs 1 and 2.

Remark 4.2: In the absence of uncertainty in the input matrix, that is for $\Delta B = 0$, we have $V = 0$ and thus $a = 0$. This corresponds to a gain margin of $[1/2, \infty)$, as in the case of the linear optimal regulator design. □

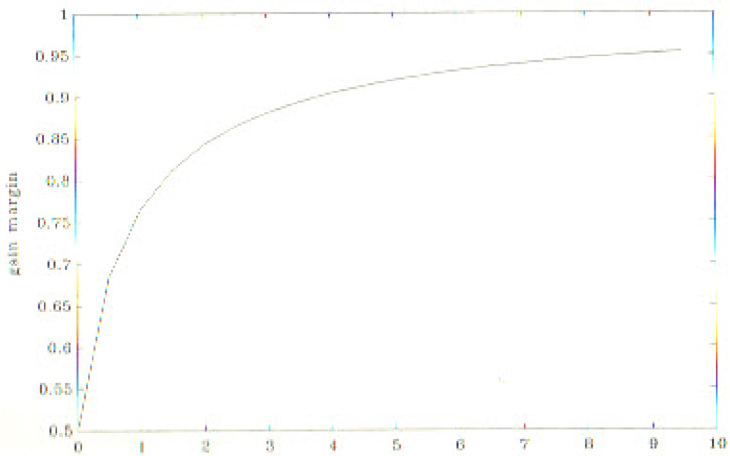

Figure 1. Minimal gain margin value as a function of a.

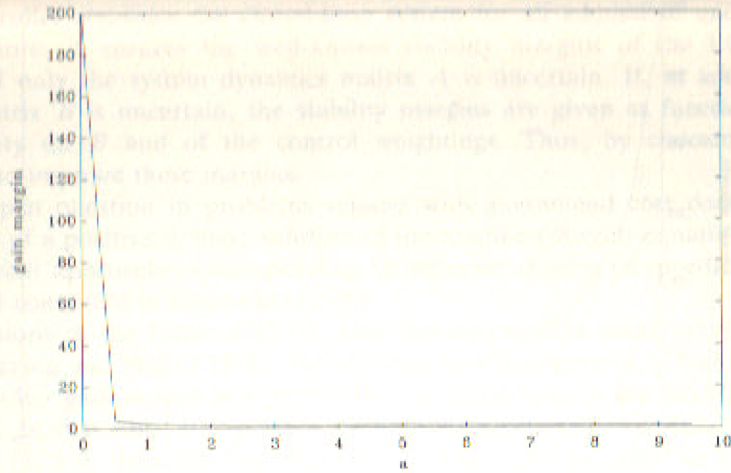

Figure 2. Maximal gain margin value as a function of a.

4.2. *Phase variation*

Consider now the case in which the unstructured uncertainty corresponds to a phase variation, i.e. $\hat{L}(s) = \hat{L} = \exp(j\phi)I$. Assuming $R = \rho I$ $\rho > 0$, we obtain $H(s) = \rho^{\frac{1}{2}}\exp(j\phi)\,\rho^{-\frac{1}{2}} = \exp(j\phi)I = \hat{L}$, $X = \rho^{-\frac{1}{2}}V\rho^{-\frac{1}{2}} = V/\rho$, $J = \exp(j\phi)I - I$. Let $a = \bar{\sigma}(V)/\rho$ as in (4.10). Due to the fact that

$$2\mu(H) = \lambda_{\min}[(\exp(j\phi) + \exp(-j\phi))I] = 2\cos\phi$$

from Theorem 4.1 one obtains the stability condition

$$2\cos\phi \geqslant 1 + [\bar{\sigma}(V)/\rho]\{(|\exp(j\phi) - 1| + 1)^2 - 1\} \tag{4.13}$$

The following Corollary can be demonstrated:

Corollary 4.2—Phase margin: *The phase margin PM of the closed-loop system obtained by the guaranteed cost controller is given by the expression*

$$PM = 2\arcsin\{2[(a^2 + a + 1)^{\frac{1}{2}} + a]\}^{-1} \tag{4.14}$$

Proof: The Proof is given in the Appendix. □

Remark 4.3: In the absence of uncertainty in the input matrix, i.e. for $\Delta B = 0$, it is $V = 0$ and thus $a = 0$. In this case, the phase margin is $PM = 2\arcsin(\frac{1}{2}) = 60°$. In other words, if only the system dynamics matrix is uncertain, the guaranteed cost controller ensures the same phase margin as the linear optimal regulator. It should also be noted that the value of a is critical in expression (4.14). Phase margin as a function of a is given in Fig. 3. □

5. Extension to the a-stability case

The results of previous sections will now be extended to the a-stability case.

Roughly speaking, a system (Σ) is stable if, in the absence of any external perturbation, and whatever the initial condition x_0 is, the state $x(t) \to 0$, as the time $t \to \infty$. If we wish (Σ) to have a time constant $\tau \leqslant \tau'$, $\tau' > 0$, we must chose the control so that $\exp(at)x(t) \to 0$ as $t \to \infty$, where $a = 1/\tau'$. In this case, (Σ) is said to be a-stable. A strict definition of a-stability is given in (Bourlès 1986,

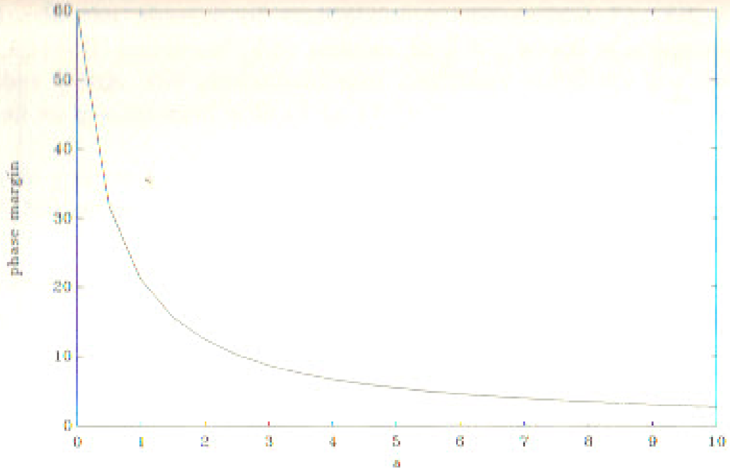

Figure 3. Phase margin as a function of a.

1987). The sufficient condition for stability in presence of unstructured uncertainty, given by Bourlès (1986), has the form

$$[\hat{T}(s) + aI]^* P + P[\hat{T}(s) + aI] \leqslant -Q \tag{5.1}$$

$\forall s \in I_a = \{s\colon \mathrm{Re}(s) = -a\}$. For the system described in § 4 it is

$$\hat{T}(s) = (A_0 + \Delta A) + (B_0 + \Delta B)\hat{L}(s)K \tag{5.2}$$

Besides, since a-stability is required, the guaranteed cost equation (2.13) will have the form

$$(A_0 + aI)^{\mathrm{T}} P + P(A_0 + aI) - PB_0R^{-1}B_0^{\mathrm{T}}P + PB_0R^{-1}VR^{-1}B_0^{\mathrm{T}}P + PWP + PTP + U + Q = 0 \tag{5.3}$$

Sufficient condition for a-stability of the guaranteed cost control is given by means of the following Theorem.

Theorem 5.1: *The closed-loop system* (2.1)′, (3.2)′ *remains a-stable in the presence of unstructured uncertainty* $\hat{L}(s)$ *if the following condition is satisfied*

$$2 \inf_{s \in I_a} \mu[H(s)] \geqslant 1 + \bar{\sigma}(X)\{[\|J(s)\|_{a,\infty} + 1]^2 - 1\} \tag{5.4}$$

$\forall s \in I_a$, where $H(s)$, $X(s)$ and $J(s)$ are defined by (4.7), (4.8) and (4.9), respectively.

Proof: The proof is given in the Appendix. □

Condition (5.4) is the same as condition (4.6) of Theorem 4.1. Hence, it can be easily shown that the gain and phase margins corresponding to a-stability are similar to the stability margins (4.12) and (4.14).

6. Discussion and concluding remarks

In previous sections a stability condition is given for uncertain systems characterized by both, structured and unstructured uncertainties, and controlled by the guaranteed cost controller. If only structured uncertainties are present, this controller stabilizes the closed-loop system for all admissible uncertainties. Furthermore, it ensures the well-known stability margins of the LQ optimal design, if only the system dynamics matrix A is uncertain. If, in addition, the input matrix B is uncertain, the stability margins are given as functions of the uncertainty on B and of the control weightings. Thus, by choosing R it is possible to improve these margins.

An open question in problems related with guaranteed cost control is the existence of a positive definite solution of the modified Riccati equation.

Different approaches corresponding to different choices of specific bounding terms are compared in Kosmidou (1990).

Equations of the form of (2.13) have been treated in some previous works (e.g. Petersen and Hollot 1986), by introducing a parameter $\varepsilon > 0$ which divides the controller gain as well as a part of the quadratic term of the modified Riccati equation. In that case, if a positive definite solution exists for some $\varepsilon = \varepsilon_1 > 0$ and $Q = Q_1 > 0$, then for any other $Q_2 > 0$ there always exists an $\varepsilon_2 > 0$ such that a positive definite solution also exists. However, one needs to try several choices of Q and gradually vary ε until finding a solution or declare failure of the process.

Existence conditions which can be exploited by the design procedure are related to the so-called 'matching conditions' (Barmish Corless and Leitmann 1983). It is easily shown that the standard positivity condition of the quadratic term of the modified Riccati equation (2.13), implies matching condition.

Claim 6.1: Let

$$M = B_0R^{-1}B_0^{\mathrm{T}} - B_0R^{-1}VR^{-1}B_0^{\mathrm{T}} - W - T \tag{6.1}$$

If $M > 0$, then the matching conditions

$$\Delta A(r) = B_0E(r), \quad \Delta B = B_0D(q) \tag{6.2}$$

are satisfied. □

Proof: The proof is given in the Appendix. □

On the other hand, the matching conditions allow to ensure the positivity of the quadratic term.

Claim 6.2: *Let* $\Delta A(r) = B_0E(r)$, $\Delta B(q) = B_0D(q)$, *and* $R = \rho I$, $\rho > 0$. *The matrix*

$$M = B_0R^{-1}B_0^{\mathrm{T}} - B_0R^{-1}VR^{-1}B_0^{\mathrm{T}} - W - T \tag{6.1$'$}$$

is positive definite if the uncertainties satisfy the condition

$$\bar{\sigma}(V)\bar{\sigma}(\tilde{W} + \tilde{T}) < \tfrac{1}{4} \tag{6.3}$$

where $\tilde{W} = B_0WB_0^{\mathrm{T}}$, $\tilde{T} = B_0TB_0^{\mathrm{T}}$ *and if* ρ *is such that*

$$\{1 - [1 - 4\bar{\sigma}(V)\bar{\sigma}(\tilde{W} + \tilde{T})]^{\frac{1}{2}}\}/2\bar{\sigma}(\tilde{W} + \tilde{T}) < \rho < \{1 + [1 - 4\bar{\sigma}(V)\bar{\sigma}(\tilde{W} + \tilde{T})]^{\frac{1}{2}}\}/2\bar{\sigma}(\tilde{W} + \tilde{T}) \tag{6.4}$$

Proof: The proof is given in the Appendix. □

In other words, a positive definite solution to (2.13) can be found by choosing the control weighting matrix consistent with (6.4). In the particular case where only the systems dynamics matrix A is uncertain, i.e. for $\Delta B = 0$, (6.4) reduces to

$$0 < \rho < 1/\bar{\sigma}(\tilde{T}) \tag{6.5}$$

while condition (6.3) vanishes. However, for $\Delta B \neq 0$, (6.3) can be conservative.

Less conservative conditions under which (2.13) has a positive definite symmetric solution are given in (Kosmidou *et al*. 1991). These results are stated in terms of the eigenvalues of the associated Hamiltonian matrix.

Besides, in the case where the matching conditions are not satisfied, one can find matrices $E = B_0^+ \Delta A$ and $D = B_0^+ \Delta B$ such that the norms $\|\Delta A - B_0 E\|$ and $\|\Delta B - B_0 D\|$ are minimized, where $(\cdot)^+$ denotes the pseudo-inverse of matrix $(\cdot)$. These matrices E and D have the meaning of orthogonal projections of ΔA and ΔB on Im B_0.

Let $E' = \Delta A - B_0 E$, $D' = \Delta B - B_0 D$. The system (2.1) obtains the form

$$\dot{x}(t) = (A_0 + B_0 E)x(t) + (B_0 + B_0 D)u(t) + f(x, u) \tag{6.6}$$

where $f(x, u) = E'x(t) + D'u(t)$.

Thus, uncertainty has been decomposed in two parts, one matched and one mismatched. A guaranteed cost controller can always be found to stabilize the matched part. As shown by Bourlès (1986), if, in addition, the condition

$$\|f(x, u)\|/\|x\| \leq 1/[2\|P\|\,\|Q^{-1}\|] \tag{6.7}$$

is satisfied, the same controller will also stabilize the mismatched part.

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Appendix

Proof of Lemma 4.1:

(a) For inequality (4.4) we have

$$2x^{\mathrm{T}} P \Delta A(r) x = 2x^{\mathrm{T}} P \sum_{i=1}^{k} A_i r_i x \leq \left| 2x^{\mathrm{T}} P \sum_{i=1}^{k} A_i r_i x \right| \leq \sum_{i=1}^{k} |2x^{\mathrm{T}} P A_i r_i x|$$

$$\leq \bar{r} \sum_{i=1}^{k} |2x^{\mathrm{T}} P d_i e_i^{\mathrm{T}} x| \leq x^{\mathrm{T}} P \bar{r} \sum_{i=1}^{k} d_i d_i^{\mathrm{T}} P x + x^{\mathrm{T}} \bar{r} \sum_{i=1}^{k} e_i e_i^{\mathrm{T}} x$$

$$= x^{\mathrm{T}} PTPx + x^{\mathrm{T}} Ux \tag{A 1}$$

where the property $|2\alpha\beta| \leq \alpha^2 + \beta^2$ has been used and T, U are defined by (2.8).

(b) For inequality (4.5) we have

$$-x^* P \Delta B \hat{L}(s) R^{-1} B_0^{\mathrm{T}} P x \leq |x^* P \Delta B \hat{L}(s) R^{-1} B_0^{\mathrm{T}} P x|$$

$$= \left|x^* P \sum_{i=1}^{l} B_i q_i \hat{L}(s) R^{-1} B_0^{\mathrm{T}} P x\right| \leq \sum_{i=1}^{l} |x^* P B_i q_i \hat{L}(s) R^{-1} B_0^{\mathrm{T}} P x|$$

$$\leq \bar{q} \sum_{i=1}^{l} |x^* P B_i \hat{L}(s) R^{-1} B_0^{\mathrm{T}} P x| = \bar{q} \sum_{i=1}^{l} |x^* P f_i g_i^{\mathrm{T}} \hat{L}(s) R^{-1} B_0^{\mathrm{T}} P x|$$

$$\leq (\tfrac{1}{2}) \bar{q} \sum_{i=1}^{l} x^* P f_i f_i^{\mathrm{T}} P x + (\tfrac{1}{2}) \bar{q} \sum_{i=1}^{l} x^* P B_0 R^{-1} \hat{L}^*(s) g_i g_i^{\mathrm{T}} \hat{L}(s) R^{-1} B_0^{\mathrm{T}} P x$$

$$= (\tfrac{1}{2}) x^* [PWP + PB_0 R^{-1} \hat{L}^*(s) V \hat{L}(s) R^{-1} B_0^{\mathrm{T}} P] x \tag{A 2}$$

where W, V are defined by (2.8). Similarly,

$$-x^* P B_0 R^{-1} \hat{L}^*(s) \Delta B^{\mathrm{T}} P x \leq (\tfrac{1}{2}) x^* [PWP + PB_0 R^{-1} \hat{L}^*(s) V \hat{L}(s) R^{-1} B_0^{\mathrm{T}} P] x \tag{A 3}$$

and then (4.5) follows. □

Proof of Theorem 4.1: Using (2.12)′ as well as (4.4) and (4.5) the sufficient condition for stability (4.1) is written

$$A_0^{\mathrm{T}} P + PA_0 - PB_0 R^{-1} \hat{L}^*(s) B_0^{\mathrm{T}} P - PB_0 \hat{L}(s) R^{-1} B_0^{\mathrm{T}} P + Q + PTP + U$$
$$+ PWP + PB_0 R^{-1} \hat{L}^*(s) V \hat{L}(s) R^{-1} B_0^{\mathrm{T}} P \leq 0 \tag{A 4}$$

$\forall s \in I_0$. From (2.13) we have

$$A_0^{\mathrm{T}} P + PA_0 + P(T + W)P + U + Q = PB_0(R^{-1} - R^{-1} V R^{-1}) B_0^{\mathrm{T}} P \tag{A 5}$$

Thus, (A 4) becomes

$$PB_0[R^{-1} + R^{-1} \hat{L}^*(s) V \hat{L}(s) R^{-1} - R^{-1} V R^{-1} - R^{-1} \hat{L}^*(s) - \hat{L}(s) R^{-1}] B_0^{\mathrm{T}} P \leq 0$$

or equivalently

$$R^{-1} \hat{L}^*(s) V \hat{L}(s) R^{-1} - R^{-1} \hat{L}^*(s) - \hat{L}(s) R^{-1} \leq R^{-1} V R^{-1} - R^{-1} \tag{A 6}$$

$\forall s \in I_0$. Right and left multiplication of (A 6) by $R^{\frac{1}{2}}$ yields

$$R^{-\frac{1}{2}} \hat{L}^*(s) V \hat{L}(s) R^{-\frac{1}{2}} - R^{-\frac{1}{2}} \hat{L}^*(s) R^{\frac{1}{2}} - R^{\frac{1}{2}} \hat{L}(s) R^{-\frac{1}{2}} \leq R^{-\frac{1}{2}} V R^{-\frac{1}{2}} - I \tag{A 7}$$

$\forall s \in I_0$. Setting

$$H(s) := R^{\frac{1}{2}} \hat{L}(s) R^{-\frac{1}{2}} \tag{A 8}$$

and

$$X := R^{-\frac{1}{2}} V R^{-\frac{1}{2}} \tag{A 9}$$

(A 7) can be written as

$$H^*(s) X H(s) - H^*(s) - H(s) \leq X - I$$

or

$$H^*(s) + H(s) \geq I + H^*(s) X H(s) - X \tag{A 7′}$$

In other words, the sufficient condition for stability obtains the form (A 7′). Since

$$\bar{\sigma}[H^*(s) X H(s) - X] I \geq H^*(s) X H(s) - X, \quad \forall s \in I_0,$$

$\bar{\sigma}(\cdot)$ denoting the maximum singular value of matrix $(\cdot)$, (A 7′) is satisfied, if

$$H^*(s) + H(s) \geq I + \bar{\sigma}[H^*(s) X H(s) - X] I \tag{A 10}$$

Let

$$J(s) := H(s) - I \tag{A 11}$$

Then

$$\begin{aligned} H^*(s)XH(s) - X &= [I + J(s)]^* X[I + J(s)] - X \\ &= J^*(s)X + XJ(s) + J^*(s)XJ(s) \end{aligned}$$

and thus

$$\begin{aligned} \bar{\sigma}[H^*(s)XH(s) - X] &\leqslant 2\bar{\sigma}(X)\|J\|_\infty + \|J\|_\infty\bar{\sigma}(X)\|J\|_\infty = \bar{\sigma}(X)\{2\|J\|_\infty + \|J\|_\infty^2\} \\ &= \bar{\sigma}(X)\{[\|J\|_\infty + 1]^2 - 1\} \end{aligned}$$

Consequently, (A 10) is satisfied, if

$$H^*(s) + H(s) \geqslant I + \bar{\sigma}(X)\{[\|J\|_\infty + 1]^2 - 1\}I \tag{A 12}$$

Moreover, it is well known that

$$\{\lambda_{\min}[H^*(s) + H(s)]\}I \leqslant H^*(s) + H(s) \tag{A 13}$$

By using (A 12), (A 13) the sufficient condition for stability obtains the form

$$\lambda_{\min}[H^*(s) + H(s)] \geqslant 1 + \bar{\sigma}(X)\{[\|J\|_\infty + 1]^2 - 1\} \tag{A 14}$$

Hence, by setting

$$\lambda_{\min}[H^*(s) + H(s)] = 2\mu(H)$$

we obtain

$$2 \inf_{s \in I_0} \mu[H(s)] \geqslant 1 + \bar{\sigma}(X)\{[\|J\|_\infty + 1]^2 - 1\} \tag{A 15}$$

□

Proof of Corollary 4.1: Condition (4.11) will be investigated in the cases (i) $g > 1$ and (ii) $g < 1$.

(i) Case $g > 1$. In this case (4.11) becomes

$$2g \geqslant 1 + a(g^2 - 1) \tag{A 16}$$

It can be easily shown that (A 16) is equivalent to

$$1 < g \leqslant (1/a)\{1 + [1 - a(1 - a)]^{\frac{1}{2}}\} \tag{A 17}$$

(ii) Case $g < 1$. In this case the stability condition (4.11) becomes

$$2g \geqslant 1 + a[(2 - g)^2 - 1] \tag{A 18}$$

which is equivalent to

$$(1/a)\{2a + 1 - [1 + a(3 + a)]^{\frac{1}{2}}\} < g < 1 \tag{A 19}$$

From (A 17), (A 19) we obtain the allowable variations of g in terms of gain margin (4.12). □

Proof of Corollary 4.2: An alternative form for condition (4.13) is

$$2[1 - 2\sin^2(\phi/2)] \geqslant 1 + a\{[2\sin(\phi/2) + 1]^2 - 1\} \tag{A 20}$$

This condition will be investigated in two cases (i) $\sin(\phi/2) \geqslant 0$ and (ii) $\sin(\phi/2) < 0$.

(i) Case $0 \leqslant \sin(\phi/2) \leqslant 1$. In this case (A 20) becomes

$$2[1 - 2\sin^2(\phi/2)] \geqslant 1 + a\{[2\sin(\phi/2) + 1]^2 - 1\} \tag{A 21}$$

It can be easily shown that this is equivalent to

$$0 \leqslant \sin(\phi/2) \leqslant [-a + (a^2 + a + 1)^{\frac{1}{2}}][2(a + 1)]^{-1} \tag{A 22}$$

(ii) Case $-1 \leqslant \sin(\phi/2) < 0$. In this case (A 20) yields

$$2[1 - 2\sin^2(\phi/2)] \geqslant 1 + a\{[1 - 2\sin(\phi/2)]^2 - 1\} \tag{A 23}$$

Similarly to the previous case, it can be shown that this is equivalent to

$$[a - (a^2 + a + 1)^{\frac{1}{2}}][2(a + 1)]^{-1} \leqslant \sin(\phi/2) < 0 \tag{A 24}$$

Conditions (A 22), (A 24) yield the allowable phase variations for stability, i.e.

$$\begin{aligned} [a - (a^2 + a + 1)^{\frac{1}{2}}][2(a + 1)]^{-1} &\leqslant \sin(\phi/2) \\ &\leqslant [-a + (a^2 + a + 1)^{\frac{1}{2}}][2(a + 1)]^{-1} \end{aligned} \tag{A 25}$$

or

$$|\sin(\phi/2)| \leqslant [-a + (a^2 + a + 1)^{\frac{1}{2}}][2(a + 1)]^{-1} \tag{A 26}$$

and finally

$$|\phi/2| \leqslant \arcsin\{2[(a^2 + a + 1)^{\frac{1}{2}} + a]\}^{-1} \tag{A 27}$$

which is equivalent to (4.14). □

Proof of Theorem 5.1: Using (5.2), (4.4) and (4.5) the sufficient condition (5.1) is written

$$\begin{aligned} (A_0 + aI)^{\mathrm{T}}P + P(A_0 + aI) - PB_0R^{-1}\hat{L}^*(s)B_0^{\mathrm{T}}P - PB_0\hat{L}(s)R^{-1}B_0^{\mathrm{T}}P + PWP \\ + PTP + U + Q + PB_0R^{-1}\hat{L}^*(s)V\hat{L}(s)R^{-1}B_0^{\mathrm{T}}P \leqslant 0 \end{aligned} \tag{A 28}$$

From (5.3) we have

$$\begin{aligned} (A_0 + aI)^{\mathrm{T}}P + P(A_0 + aI) + PWP + PTP + U + Q \\ = PB_0R^{-1}B_0^{\mathrm{T}}P - PB_0R^{-1}VR^{-1}B_0^{\mathrm{T}}P \end{aligned} \tag{A 29}$$

Thus, (A 28) becomes

$$PB_0[R^{-1} + R^{-1}\hat{L}^*(s)V\hat{L}(s)R^{-1} - R^{-1}VR^{-1} - R^{-1}\hat{L}^*(s) - \hat{L}(s)R^{-1}]B_0^{\mathrm{T}}P \leqslant 0$$

or equivalently

$$R^{-1}\hat{L}^*(s)V\hat{L}(s)R^{-1} - R^{-1}\hat{L}^*(s) - \hat{L}(s)R^{-1} \leqslant R^{-1}VR^{-1} - R^{-1} \tag{A 30}$$

$\forall s \in I_a$. The above condition is the same as (A 6). Thus, as proved in Theorem 4.1, (A 30) is equivalent to condition (5.4). □

Proof of Claim 6.1: Let

$$M = B_0R^{-1}B_0^{\mathrm{T}} - B_0R^{-1}VR^{-1}B_0^{\mathrm{T}} - W - T > 0 \tag{A 31}$$

Then

$$x^{\mathrm{T}}(t)B_0R^{-1}B_0^{\mathrm{T}}x(t) - x^{\mathrm{T}}(t)[B_0R^{-1}VR^{-1}B_0^{\mathrm{T}} + W + T]x(t) > 0 \quad \text{(A 32)}$$

for all $x(t) \in \mathbb{R}^n$. It follows that $B_0^{\mathrm{T}}x(t) = 0 \Rightarrow (W + T)x(t) = 0$. Hence, $\mathrm{Ker}\, B_0^{\mathrm{T}} \subset \mathrm{Ker}(W + T)$ so that $\mathrm{Im}(W + T) \subset \mathrm{Im}\, B_0$. Therefore, $W + T$ is of the form $W + T = B_0Y$ and since W and T are symmetric matrices, it is $W + T = B_0ZB_0^{\mathrm{T}}$, or equivalently

$$\bar{r}\sum_{i=1}^{k} d_id_i^{\mathrm{T}} + \bar{q}\sum_{j=1}^{l} f_jf_j^{\mathrm{T}} = B_0ZB_0^{\mathrm{T}}.$$

By multiplying left by $x^{\mathrm{T}}(t)$ and right by $x(t)$ we have $B_0^{\mathrm{T}}x(t) = 0 \Rightarrow d_i^{\mathrm{T}}x(t) = 0$ and $f_j^{\mathrm{T}}x(t) = 0$, for all i and j. Thus, $\mathrm{Ker}\, B_0^{\mathrm{T}} \subset \mathrm{Ker}\, d_i^{\mathrm{T}}$ and $\mathrm{Ker}\, B_0^{\mathrm{T}} \subset \mathrm{Ker}\, f_j^{\mathrm{T}}$ and hence $\mathrm{Im}\, d_i \subset \mathrm{Im}\, B_0$ and $\mathrm{Im}\, f_j \subset \mathrm{Im}\, B_0$. So, d_i and f_j are respectively of the form

$$d_i = B_0\tilde{d}_i, \quad f_j = B_0\tilde{f}_j \quad \text{(A 33)}$$

By denoting

$$E(r) = \sum_{i=1}^{k} \tilde{d}_ie_i^{\mathrm{T}}r_i(t) \quad \text{and} \quad D(q) = \sum_{j=1}^{l} \tilde{f}_jg_j^{\mathrm{T}}q_j(t) \quad \text{(A 34)}$$

(2.2)–(2.7) yield

$$\Delta A(r) = B_0E(r) \quad \text{and} \quad \Delta B(q) = B_0D(q)$$

□

Proof of Claim 6.2: The matching conditions imply

$$T = \bar{r}\sum_{i=1}^{k} d_id_i^{\mathrm{T}} = \bar{r}\sum_{i=1}^{k} B_0\tilde{d}_i\tilde{d}_i^{\mathrm{T}}B_0^{\mathrm{T}} = B_0\bar{r}\sum_{i=1}^{k}\tilde{d}_i\tilde{d}_i^{\mathrm{T}}B_0^{\mathrm{T}} = B_0\tilde{T}B_0^{\mathrm{T}} \quad \text{(A 35)}$$

$$W = \bar{q}\sum_{j=1}^{l} f_jf_j^{\mathrm{T}} = \bar{q}\sum_{j=1}^{l} B_0\tilde{f}_j\tilde{f}_j^{\mathrm{T}}B_0^{\mathrm{T}} = B_0\bar{q}\sum_{j=1}^{l}\tilde{f}_j\tilde{f}_j^{\mathrm{T}}B_0^{\mathrm{T}} = B_0\tilde{W}B_0^{\mathrm{T}} \quad \text{(A 36)}$$

Hence,

$$\begin{aligned} M &= B_0R^{-1}B_0^{\mathrm{T}} - B_0R^{-1}VR^{-1}B_0^{\mathrm{T}} - W - T \\ &= B_0[R^{-1} - R^{-1}VR^{-1} - \tilde{W} - \tilde{T}]B_0^{\mathrm{T}} \end{aligned} \quad \text{(A 37)}$$

and since $R = \rho I$, M is positive definite for

$$I/\rho - V/\rho^2 - \tilde{W} - \tilde{T} > 0 \quad \text{(A 38)}$$

This condition is obviously satisfied if (6.3), (6.4) are satisfied. □

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