Comment Regarding "On Delay-Independant Stability of Large-Scale Systems with Time Delays"

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Abstract—**This paper comments the result of a recent paper.** [1] **We point out that these results were already published in a more general framework [1], some points of which are also clarified: the state which has to be considered and a connection with the passivity theorem.**

I. Introduction

In [1], the "α-stability" of a large-scale interconnected system described by functional differential equations is studied. As a matter of fact, the "asymptotic α-stability" is defined for such a system, and the usual asymptotic stability, which is considered in the paper,[1] corresponds to the case $\alpha = 0$. Roughly speaking, a system with state $X: t \to X(t)$ is asymptotically α-stable if and only if the system with state $t \to e^{\alpha t} X(t)$ is asymptotically stable. The connection between the result of the above-mentioned paper[1] and the main result of [1] is pointed out in this paper. In addition, some points of [1] (such as the state which has to be considered) are also clarified. Finally, the connection between the main result of [1] and the passivity theorem is shown.

II. Restatement of the Main Result of [1]

Let us first recall the basic result of [1] in the case $\alpha = 0$ and clarify some points. Let h be a nonnegative real number, and let $\mathcal{X}_i, 1 \le i \le n$ be Hilbert spaces (e.g., $\mathcal{X}_i = \mathbb{R}^{n_i}$); $L_2(\mathcal{X}_i)$ denotes the Hilbert space of functions $x_i: [-h, \infty) \to \mathcal{X}_i$ which are Lebesgue-measurable and square integrable on $[-h, \infty)$ (this space is equipped with the standard inner product). Set $X = \Pi_{i=1}^n \mathcal{X}_i$; if $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ belong to $L_2(\mathcal{X})$, the inner product $\langle x, y \rangle$ is defined as $\langle x, y \rangle = \Sigma_{i=1}^n \langle x_i, y_i \rangle$, where $\langle x_i, y_i \rangle$ is the inner product of x_i and y_i in $L_2(\mathcal{X}_i)$. In all Hilbert spaces above, the norm is defined from the inner product as usually.

The truncation operator P_t to the interval $[0, t]$ is used. For any function x defined on $[-h, \infty), (P_t x)(\tau) = x(\tau)$ if $\tau \in [0, t]$ and $(P_t x)(\tau) = 0$ otherwise. If $\mathcal{Y}$ is any Hilbert space, the "extended space" $L_{2e}(\mathcal{Y})$ is the space of functions $y: [-h, \infty) \to \mathcal{Y}$ such that $\forall t \in \mathbb{R}^+$, the restriction of y to the interval $[-h, t]$ belongs to $L_2(\mathcal{Y})$. For any functions y and z belonging to $L_{2e}(\mathcal{Y})$ and any $t \in \mathbb{R}^+$, we set $\langle y, z \rangle_t = \langle P_t y, P_t z \rangle$ and $\|y\|_t = \|P_t y\|$ (i.e., the norm of $P_t y$ in $L_2(\mathcal{Y})$).

The system Σ under consideration in [1] is defined by n coupled functional differential equations

$$\dot{x}_i(t) = F_i(x_i)(t) + H_i(x)(t), \qquad t \ge 0 \tag{1}$$

where F_i is a causal operator on $L_{2e}(\mathcal{X}_i)$ to $L_{2e}(\mathcal{X}_i)$ and H_i is a causal operator on $L_{2e}(X)$ to $L_{2e}(\mathcal{X}_i)$, with $F_i(0) = H_i(0) = 0$.

In addition, assume that $F_i(x_i)(t)$ and $H_i(x)(t)$ only depend on the restrictions of x_i and x, respectively, to $[t-h, t]$. The state $X(t)$ is defined as the function $\tau \to x(t+\tau), \tau \in [-h, 0]$, and its norm is defined by $\|X(t)\|^2 = \|x(t)\|^2 + \int_{-h}^0 \|x(t+\tau)\|^2 \, d\tau)$ (see, e.g., [6]). The initial state $X(0)$ is denoted by X_0. It is assumed that once X_0 is fixed, the functions x_i and $x = (x_1, \cdots, x_n)$ are well defined and absolutely continuous on $\mathbb{R}^+$ so that they belong to $L_{2e}(\mathcal{X}_i)$ and $L_{2e}(X)$, respectively. Notice that $X = 0$ is an equilibrium point of Σ.

It is assumed that the operators F_i and H_i are finite-gain L_2-stable, i.e., there exist nonnegative constants μ_i and $\mu_{ij}, 1 \le i \le n, 1 \le j \le n$ such that for every x_i in $L_{2e}(\mathcal{X}_i)$, every $x = (x_1, \cdots, x_n)$ in $L_{2e}(X)$, and every t in $\mathbb{R}^+$

$$\|F_i(x_i)\|_t \le \mu_i \|x_i\|_t + \alpha_i \tag{2}$$

$$\|H_i(x)\|_t \le \sum_{i=1}^n \mu_{ij} \|x_j\|_t + \beta_i \tag{3}$$

where α_i and β_i are functions of $\|X_0\|$ and $\beta_i(\|X_0\|)$ tends to zero as $\|X_0\|$ tends to zero (whereas $\alpha_i(\|X_0\|)$ remains bounded).[2]

Let $\mathcal{L}(\mathcal{X}_i)$ denote the space of linear continuous mappings $Q_i: \mathcal{X}_i \to \mathcal{X}_i; Q_i \ge 0$ means that Q_i is self-adjoint and nonnegative and $Q_i \gg 0$ means that there exists $\varepsilon > 0$ such that $Q_i - \varepsilon I_i \ge 0$ (where I_i is the identity of $\mathcal{X}_i$).

The main result of [1, Th. 2] can be stated again as follows (in the case $\alpha = 0$).

Theorem: Assume the following.

i) For every $i, 1 \le i \le n$, there exists (in $\mathcal{L}(\mathcal{X}_i)$) $P_i \ge 0$ and $Q_i \gg 0$ such that for every x_i in $L_{2e}(\mathcal{X}_i)$ and every t in $\mathbb{R}^+$

$$\langle x_i, P_i F_i(x_i) + \tfrac{1}{2} Q_i x_i \rangle_t + \gamma_i \le 0 \tag{4}$$

where γ_i is a function of $\|X_0\|$ such that $\gamma_i(\|X_0\|)$ tends to zero as $\|X_0\|$ tends to zero.[3]

ii) C is an M-matrix, where $C = (c_{ij})_{n \times n}$ is defined by

$$c_{ij} = \delta_{ij} - 2\mu_{ij} \|P_i\| \|Q_i^{-1}\|$$

where δ_{ij} is the Kronecker symbol. Then, the equilibrium point zero of Σ is globally asymptotically stable.

Remark: This is only one part of [1, Th. 2] where an input–output stability result is also obtained.

III. A Particular Case

The result[1] corresponds to the case where each Hilbert space $\mathcal{X}_i$ is finite dimensional and where

$$F(x_i)(t) = A_i x_i(t)$$

$$H_i(x)(t) = \sum_{i=1}^n A_{ij} x_j(t - h_{ij}(t))$$

where $0 \le h_{ij}(t) \le h$.

Conditions (2) and (3) are clearly satisfied with $\mu_i = \overline{\sigma}(A_i)$ and $\mu_{ij} = \overline{\sigma}(A_{ij})$. Condition i) (with an equality and $\gamma_i = 0$) reduces to the Lyapunov equation

$$A_i^T P_i + P_i A_i + Q_i = 0.$$

In other words, A_i is required to have eigenvalues with negative real parts, as in the paper.[1]

Finally, Condition ii) is clearly equivalent to the one given in the paper.[1]

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[1] B. Xu, *IEEE Trans. Automat. Contr.*, vol. 40, pp. 930–933, 1995.

[2] In the original result, $\beta_i(\|X_0\|)$ is assumed to be zero, but it is easy to verify that the theorem cited below is still true (see its proof in [1]) with the more general assumption considered here.

[3] In the original result, $\gamma_i(\|X_0\|)$ is assumed to be zero, but it is easy to verify that the theorem is still true with this more general assumption.

IV. Conclusion

As a conclusion, note that [1, Th. 2] has been applied to the case of linear delay systems in [2]. Note also that this theorem is connected with the well-known *passivity theorem* when the interconnections are zero, i.e., $H_i = 0, i = 1, \cdots, n$. The ith subsystem Σ_i defined by (1) can be viewed as defined by the operator G_{i1}: $z_i \to x_i$, with $x_i(t) = P_i^{-1} \int z_i(t)\, dt$ fed back by the operator $G_{i2} = -P_i F_i$ (with input x_i and output $-z_i$); G_{i1} is passive. By (4), G_{i2} is strictly passive, and by (2) it is finite-gain L_2-stable. Hence, x_i belongs to $L_2(\mathcal{X}_i)$ (see, e.g., [7]). The proof of the global attractivity of zero is then straightforward: $\int_{-h}^{0} \|x_i(t+\tau)\|^2\, d\tau$ tends to zero as t tends to infinity. In addition, F_i is finite-gain stable; therefore, $\dot{x}_i$ belongs to $L_2(\mathcal{X}_i)$. This proves that $\|x_i(t)\|$ tends to zero as t tends to infinity, hence so does $\|X_i(t)\|$, i.e., zero is globally attractive for Σ_i. For proving that in addition zero is Lyapunov-stable for Σ_i, see [3] and use the definition of the state given above (at this step, it is necessary to use the fact that $\alpha_i(\|X_0\|)$ tends to zero as $\|X_0\|$ tends to zero).

The condition ii) of the theorem means that the interconnections are weak enough for the large-scale system Σ to remain asymptotically stable.

For a discrete-time version of this theorem in the case where the interconnections are zero, see [4] and [5]; it is connected with the *small gain theorem.*

References

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