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An intrinsic algebraic setting for poles and zeros of linear time-varying systems

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ABSTRACT

In this paper, poles and zeros are defined for linear time-varying systems using suitable ground field extensions. The definitions of the system poles, transmission poles, invariant zeros, hidden modes, etc, are given in an intrinsic module-based framework and are consistent in the sense that the poles are connected to the stability of the system and the zeros to the zeroing of the output for non zero inputs. In particular, it is proved that the necessary and sufficient condition for a continuous-time system to be exponentially stable is similar to the well-known condition in the time-invariant case.

1. Introduction

The concept of poles of a system has received special attention as a means to evaluate stability. Although this relation is clear and well-known for linear time-invariant (LTI) systems, the time-varying case is much more difficult since the frozen-time (pointwise-in-time) eigenanalysis may not contain information about stability. Several definitions of poles of a linear time-varying (LTV) system have been proposed. In [17], Lyapunov transformations were used to obtain an upper triangular form of the state matrix, from which the notion of *pole set of the state matrix* was defined. In other approaches, factorizations of the differential operator associated with the system are used to evaluate stability ([12,20] and related references). Recent investigations have been carried on the polynomials over skew fields. In [18] it is shown that a skew polynomial can be written as a product of *elementary factors* $(\partial - \gamma_i)^{d_i}$ after a well-chosen (large enough) field extension while in [14] and related references the so-called Wedderburn polynomials (W-polynomials) are studied as the class of skew polynomials which factorize in distinct first order elementary factors. In the work presented here, *independent* first order factors are found for a given skew polynomial $P(X)$ using a suitable field extension. They form what is called in the sequel a *fundamental set* of those factors and it is shown that this is related to a fundamental set of solutions of the differential equation associated with the differential operator $P(\partial)$, $\partial = \mathrm{d}/\mathrm{d}t$ deduced from the skew polynomial $P(X)$. This allows us to give intrinsic definitions of the poles and zeros of LTV systems in the algebraic framework initiated by Malgrange [16] and popularized in systems theory by Fliess (see [6] and related references) in which an LTV system is modeled by a finitely presented module over a ring of linear differential operators.

The paper is organized as follows: Section 2 introduces the mathematical background, especially facts about field extensions, skew polynomials and differential operators. The different types of differential equations which may characterize an LTV system are also recalled. In Section 3 it is shown how first order factors can be obtained for a given skew polynomial using a well-chosen field extension. The latter field extension is shown to be a Picard–Vessiot one in Section 4 in which a fundamental set of solutions and independent roots are computed from the first order factors found in Section 3. The independent roots are not unique and their classes of conjugacy are studied in Section 5. The poles of an LTV system are introduced in Section 6 in a module-based formulation and the their relation to stability is explained. In the same manner, modules for the other kinds of poles and zeros are introduced in Section 7. A physical example is presented in Section 8. The extension of this approach to the nonlinear case is discussed in Section 9, while Section 10 is devoted to concluding remarks.

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2. Notations and mathematical background

2.1. Skew polynomials, differential fields and operators

In everything that follows, all fields are commutative and of characteristic zero. Let $\mathbf{K} = \mathbb{C}((t))$ be the (commutative) field of formal Laurent series with coefficients in $\mathbb{C}$, the field of complex numbers. An element a of this field is of the form $a = \sum_{i\geq\nu} c_i t^i, \quad c_i \in \mathbb{C}$. A field $\mathbf{L}$ is a *field extension* of $\mathbf{K}$ (written $\mathbf{L}/\mathbf{K}$) if $\mathbf{K} \subseteq \mathbf{L}$ is a subfield. Field extensions provide *splitting fields* for given polynomials, i.e., fields over which those polynomials *split* or, equivalently, are fully decomposable into products of first order factors. A *Galois extension* $\mathbf{L}/\mathbf{K}$ is an algebraic field extension for which every irreducible polynomial in the ring $\mathbf{K}[X]$ of polynomials in X with coefficients in $\mathbf{K}$ admitting a zero in $\mathbf{L}$ splits in $\mathbf{L}$. Every Galois extension of $\mathbf{K}$ over $\mathbb{C}$ is of the form $\mathbf{K}_m = \mathbb{C}((z)), z^m = t, z = t^{1/m}, m \in \mathbb{N}$ [18]. Moreover, $\mathbf{K}_m$ has degree m over $\mathbf{K}$ and the union $\overline{\mathbf{K}} = \bigcup_m \mathbf{K}_m$ is an algebraic closure of $\mathbf{K}$ (note that $\mathbf{K}_m \subseteq \mathbf{K}_n$ for $m \leq n$).

Let $\mathbf{k}$ be a differential field equipped with the usual derivation. Let $\mathbf{k}[\partial]$ be the ring of left differential operators in $\partial = \mathrm{d}/\mathrm{d}t$ with coefficients in $\mathbf{k}$, i.e., of all elements of the form $P(\partial) = \sum_{i=0}^{n} a_i \partial^{n-i}, a_i \in \mathbf{k}$. It is equipped with the commutation rule $\partial a = a\partial + \mathrm{d}a/\mathrm{d}t$ which is the Leibniz rule of derivation of a product. Let $\mathbf{k}[X;\delta]$, $\delta = \mathrm{d}/\mathrm{d}t$ be the ring of *skew* polynomials with indeterminate X and coefficients in $\mathbf{k}$. The multiplication in $\mathbf{k}[X;\delta]$ obeys to the commutation rule $Xa = aX + \mathrm{d}a/\mathrm{d}t$. One can associate with the skew polynomial ring $\mathbf{k}[X;\delta]$ the ring of differential operators $\mathbf{k}[\partial]$ by associating to the indeterminate X the derivative operator ∂. The two rings are isomorphic, thus are identified. The following *evaluation theory of skew polynomials* was introduced in [13]: Let $\mathbf{k}[X;\delta] \ni P(X) = \sum_{i=0}^{n} a_i X^i$ be a skew polynomial. The *evaluation of P in $\alpha \in \mathbf{k}$*, denoted by $P(\alpha)$ is defined as follows: $P(\alpha) = \sum_{i=0}^{n} a_i N_i(\alpha)$ where N_i is defined inductively by $N_0(\alpha) = 1$, $N_i(\alpha) = N_{i-1}(\alpha)\alpha + \frac{\mathrm{d}}{\mathrm{d}t}(N_{i-1}(\alpha))$. For $P \in \mathbf{k}[X;\delta]$, $\alpha \in \mathbf{k}$, $P(\alpha) = 0$ if and only if $X - \alpha$ is a right factor of $P(X)$ and, in this case, we say that α is a *(right) root* or *zero* of P. For any nonzero $c \in \mathbf{k}$, the *conjugate of α by c* denoted by α^c is $\alpha^c = \alpha + \frac{\mathrm{d}c}{\mathrm{d}t}c^{-1}$. $\Delta(\alpha) = \{\alpha^c, c \in \mathbf{k}, c \neq 0\}$ is called the *conjugacy class* of α. Let $P, G \in \mathbf{k}[X;\delta]$. A *product formula*

$$(PG)(\alpha) = 0, \quad \text{if } G(\alpha) = 0 \quad \text{or } P(\alpha^{G(\alpha)})G(\alpha), \quad \text{if } G(\alpha) \neq 0 \tag{1}$$

as well as the existence of a *left least common multiple* denoted by $[P, G]_l$ can be established for skew polynomials. $[P, G]_l$ is a monic polynomial such that $\mathbf{k}[X;\delta]P \cap \mathbf{k}[X;\delta]G = \mathbf{k}[X;\delta][P, G]_l$. There are several ways of computing the left common multiple which lead thus to several factorizations of $[P, G]_l$. Relation (1) clarifies the calculations in the case of first order factors: let $P \in \mathbf{K}[X;\delta]$ and $\alpha \in \mathbf{K}$. Then

$$[P, X - \alpha]_l = P, \quad \text{if } P(\alpha) = 0 \quad \text{or } (X - \alpha^{P(\alpha)})P, \quad \text{if } P(\alpha) \neq 0. \tag{2}$$

Let $\Delta = \{\gamma_1, \ldots, \gamma_n\} \subseteq \mathbf{k}$. The *minimal polynomial of Δ*, denoted by P_Δ is the monic generator of the left ideal $\{P \in \mathbf{k}[X;\delta],\ P(\Delta) = 0\}$: $P_\Delta = [\partial - \gamma_i, i = 1, \ldots, n]_l$. Δ is said *P-independent* if the degree of P_Δ is n. Relation (2) can be applied to construct the minimal polynomial in the general case in a recursive manner.

2.2. LTV systems

In the approach adopted by Fliess (see [6] and related references) an LTV system is a finitely presented module M over a noncommutative ring $\mathbf{R} = \mathbf{k}[\partial]$. Assuming that this system M is equipped with an input $u = (u_i)_{i=1,\ldots,m}$, i.e., a set of variables such that $[u]_\mathbf{R}$, the $\mathbf{R}$-module generated by the entries of u, is free of rank m and $M/[u]_\mathbf{R}$ is torsion, stability can be studied on the so-called *autonomous part of the system*, obtained by canceling the input variables; algebraically, we obtain in this way the torsion module $T \cong M/[u]_\mathbf{R}$. As $\mathbf{R}$ is simple, T is cyclic and can be characterized by a state-space representation $\frac{dx}{dt} = Ax, A \in \mathbf{k}^{n\times n}$ or, equivalently, by a scalar differential equation

$$P(\partial)y = 0, \qquad P(\partial) = \partial^n + \sum_{i=1}^{n} a_i \partial^{n-i} \tag{3}$$

where y is a generator of T.

3. Polynomial factorization

In [18] it is shown how a suitable field extension $\mathbf{K}_m/\mathbf{K}$ can be constructed in order to find factors of a skew polynomial belonging to $\mathbf{k}[X;\delta]$. This procedure is shown on

Example 1. Consider (3) with $P(\partial) = \partial^2 + (\frac{1}{2}t^{-1} - 2\alpha)\partial - t^{-3} - \frac{1}{2}t^{-1}\alpha + \alpha^2, \alpha \in \mathbb{R}$. The change of variable $\xi = t^{\frac{3}{2}}\partial$ transforms the initial equation into a differential equation with coefficients in the ring of formal series in $t^{1/2}$, $\mathbf{O}_2 = \mathbb{C}[[t^{1/2}]]$. Setting $P(\partial) = L(\xi)$, $L(\xi)y = 0$ and $L(\xi) = t^{-3}Q(\xi), Q(\xi) = \xi^2 + (\frac{1}{2}t^{\frac{1}{2}} - 2\alpha t^{\frac{3}{2}})\xi - 1 - \frac{1}{2}t^2\alpha + \alpha^2 t^3$ where the commutation rule for the new variable ξ is deduced from the initial one (introduced in Section 2.1), i.e., $\xi a = a\xi + t^{\frac{3}{2}}\frac{da}{dt}, a \in \mathbf{K}_2$. Therefore, the polynomial whose coefficients are the canonical images of those of $Q(\xi)$ in $\mathbf{O}_2/\pi\mathbf{O}_2$, $\pi = t^{1/2}$, is $\overline{Q}(\xi) = \xi^2 - 1$ which trivially factorizes over $\mathbb{C}$: $\overline{Q}(\xi) = F_1F_2$ with $F_1 = \xi - 1$ and $F_2 = \xi + 1$. Using Hensel's Lemma [18], a *unique* factorization $Q(\xi) = Q_1(\xi)Q_2(\xi)$ such that $\overline{Q_i} = F_i$, $i \in \{1, 2\}$ can be obtained from the factors F_1, F_2 via an iterative process in which terms of the form $\pi^k T_k$, $\pi^k S_k$ are added to the polynomials F_1 and F_2 respectively: $Q_1(\xi) = F_1 + \pi T_1 + \pi^2 T_2 + \cdots$ and $Q_2(\xi) = F_2 + \pi S_1 + \pi^2 S_2 + \cdots$. In the corrections $\pi^i T_i$ and $\pi^i S_i$ at each step, T_i and S_i are polynomials with coefficients in $\mathbb{C}$ and such that $d^0(T_i) < d^0(F_1)$ and $d^0(S_i) < d^0(F_2)$, where $d^0(.)$ denotes the degree of a skew polynomial. For this example, $T_i, S_i \in \mathbb{C}$ and the factors can be found in one step: $Q_1(\xi) = \xi - 1 - \frac{1}{2}t^{\frac{1}{2}} - \alpha t^{\frac{3}{2}}$, $Q_2(\xi) = \xi + 1 - \frac{1}{2}t^{\frac{1}{2}} - \alpha t^{\frac{3}{2}}$. Obviously, $\overline{Q_i} = F_i, i \in \{1, 2\}$ and one can easily check using the new commutation rule in ξ above that $Q(\xi) = Q_1(\xi)Q_2(\xi)$. These factors can be transformed to find a factorization of $P(\partial)$. Here, $P(\partial) = t^{-3}Q_1(t^{3/2}\partial)Q_2(t^{3/2}\partial) = t^{-3}(t^{\frac{3}{2}}\partial - 1 - \frac{1}{2}t^{\frac{1}{2}} - \alpha t^{\frac{3}{2}})(t^{\frac{3}{2}}\partial + 1 - \frac{1}{2}t^{\frac{1}{2}} - \alpha t^{\frac{3}{2}}) = t^{-3}(t^{\frac{3}{2}}\partial t^{\frac{3}{2}} - t^{\frac{3}{2}} - \frac{1}{2}t^2 - \alpha t^3)(\partial + t^{-\frac{3}{2}} - \frac{1}{2}t^{-1} - \alpha) = (\partial - t^{-\frac{3}{2}} + t^{-1} - \alpha)(\partial + t^{-\frac{3}{2}} - \frac{1}{2}t^{-1} - \alpha)$. This leads to a decomposition of P with two first order factors: $P(\partial) = (\partial - a_2)(\partial - a_1)$, where $a_1(t) = -t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} + \alpha$, $a_2(t) = t^{-\frac{3}{2}} - t^{-1} + \alpha$. $\diamond$

Generally, the polynomial $P(\partial)$ in (3) can be decomposed into *elementary factors*, namely

$$P(\partial) = (\partial - a_n)^{d_n} \ldots (\partial - a_1)^{d_1} \tag{4}$$

over a suitable extended field ($a_i \in \mathbf{K}_m$ for some m). The elementary factors are thus of order $1 (d_i = 1)$ or greater.

4. Independent roots and solutions, Picard–Vessiot extensions

The solution space of (3) has a dimension n over $\mathbb{C}$. A basis of this space is called a *fundamental set of solutions* of (3). A set $\{y_1, \ldots, y_n\}$ of particular solutions of (3) form such a basis if and only if the *Wronskian matrix* W in (5) is nonsingular. A

Picard–Vessiot extension of (3) is a field extension over which (3) has a fundamental set of solutions (see, e.g., [18]).

$$W(y_i)_{i=1}^n = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{dy_1}{dt} & \frac{dy_2}{dt} & \cdots & \frac{dy_n}{dt} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{d^{n-1}y_1}{dt} & \frac{d^{n-1}y_2}{dt} & \cdots & \frac{d^{n-1}y_n}{dt} \end{bmatrix}, \tag{5}$$

$$V(\gamma_i)_{i=1}^n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \partial(\gamma_1) & \partial(\gamma_2) & \cdots & \partial(\gamma_n) \\ \partial^2(\gamma_1) & \partial^2(\gamma_2) & \cdots & \partial^2(\gamma_n) \\ \cdots & \cdots & \cdots & \cdots \\ \partial^{(n-1)}(\gamma_1) & \partial^{(n-1)}(\gamma_2) & \cdots & \partial^{(n-1)}(\gamma_n) \end{bmatrix}.$$

A similar characterization can be given for the roots of the polynomial P. Note first that in a factorization of type (4), only a_1 is a right root (zero) of P in the sense of the definition given in Section 2.1 for skew polynomials. The following result will be used in what follows. It is a particularization to the conjugacy class of $0 \in \mathbf{k}$ of a more general property established for W-polynomials in [14] and related references.

Proposition 1. *For a set $\{y_1, \ldots, y_n\}$ in any differential field $\mathbf{k}$, let $\gamma_i = \frac{dy_i}{dt} y_i^{-1}$, $i = 1, \ldots, n$ and consider the Vandermonde matrix of $\gamma_1, \ldots, \gamma_n$ defined by V in (5) where $\partial^i(.)$ is the evaluation of the polynomial ∂^i as introduced in Section 2.1. Then*

(1) $W(y_i)_{i=1}^n = V(\gamma_i)_{i=1}^n \mathrm{diag}\{y_1, \ldots, y_n\}$.
(2) *The following are equivalent:*
  (i) *$W(y_i)_{i=1}^n$ (or, equivalently, $V(\gamma_i)_{i=1}^n$) is invertible*
  (ii) *The set $\{\gamma_1, \ldots, \gamma_n\}$ is P-independent.* ◇

For a given polynomial P of degree n we are interested in a characterization of the left ideals generated by its roots over different field extensions: how many P-independent roots exist and how they can be computed. The answer to the first question is the dual of a result given in [5] on the maximum number of conjugacy classes of roots of P:

Proposition 2. *A polynomial of degree n has at most n distinct P-independent roots.* ◇

Proof. Suppose there exist $n+1$ distinct P-independent roots $\{\gamma_1, \ldots, \gamma_{n+1}\}$ of P and let $y_i = e^{\int \gamma_i(t)dt}$, $i = 1, \ldots, n+1$. From Proposition 1, the y_i's are $\mathbb{C}$-linearly independent. Also, as the γ_i's are roots of P, $P(\partial) = P'(\partial)(\partial - \gamma_i)$ from which $P(\partial)y_i = P'(\partial)(\frac{dy_i}{dt} - \gamma_i y_i) = 0$ so y_i, $i = 1, \ldots, n+1$ are $n+1$ $\mathbb{C}$-independent solutions to $P(\partial)y = 0$ which is a contradiction to the fact that P is of degree n. ◇

Next, remark that a maximal set of P-independent roots corresponds to a fundamental set of solutions of the differential equation (3): let $y_1, \ldots, y_n$ denote these solutions in a suitable Picard–Vessiot extension of $\mathbf{k}$ and define $\gamma_i = \frac{dy_i}{dt} y_i^{-1}$. Note that γ_i are distinct. Also $W(y_i)_{i=1}^n$ is invertible, so, from Proposition 1, the γ_i's are P-independent. Moreover, as each y_i is (by the definition of γ_i) the solution of the elementary equation $(\partial - \gamma_i)y_i = 0$, it follows that γ_i is a root of P. Indeed, let $P(\partial) = R(\partial)(\partial - \gamma_i) + P(\gamma_i)$. As y_i is a solution to (3), $P(\partial)y_i = 0$ which gives $R(\partial)(\partial - \gamma_i)y_i + P(\gamma_i)y_i = 0$. As $(\partial - \gamma_i)y_i = 0$, it follows that $P(\gamma_i)y_i = 0$ from which $P(\gamma_i) = 0$ as $y_i \neq 0$.

Definition 1. Let $P \in \mathbf{k}[\partial]$ be a polynomial of degree n. A *fundamental set of roots of P* is a set $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ of right roots of P such that $P = P_\Delta$. ◇

A fundamental set of roots of P can be constructed starting from the factors of the polynomial P. The factorization (4) is obtained over a Galois extension $\mathbf{K}_m$ for P. As the derivation δ has a unique extension from $\mathbf{K}$ to $\mathbf{K}_m$, it can be shown that $\mathbf{K}_m$ is also a Picard–Vessiot field of the differential equation (3) and thus (3) has n $\mathbb{C}$-linearly independent solutions over $\mathbf{K}_m$ and P has n P-independent roots over that field. A fundamental set of roots can be found from the elementary factors of (4) in an iterative way: first, $\gamma_1 = a_1$. Next, γ_2 is solution of the equation $\gamma_2^{\gamma_2 - \gamma_1} = a_i$ with $i = 1$ if $d_1 > 1$ or $i = 2$ if $d_1 = 1$ which leads to the Riccati equation in γ_2: $\frac{d\gamma_2}{dt} + \gamma_2^2 - (a_i + \gamma_1)\gamma_2 - \frac{d\gamma_1}{dt} + \gamma_1 a_i$, $i = 1$ *or* 2. The latter Riccati equation has the particular solution $\gamma_2 = \gamma_1$ and is thus solvable by reduction to a linear first order one. Doing so in the same manner for the rest of the factors a_i (multiple or not), one obtains a fundamental set of roots of P, $\Delta = \{\gamma_1, \ldots, \gamma_n\}$. Notice that the γ_i's may not necessarily all belong to $\mathbf{K}_m$ but to a larger field $\check{\mathbf{K}}_m$ which is still a Picard–Vessiot extension of $\mathbf{K}$ but with the additional property that it allows the computation of a fundamental set of roots. It follows that:

$$\check{\mathbf{K}}_m[\partial]P = \bigcap_i \check{\mathbf{K}}_m[\partial](\partial - \gamma_i), \quad i = 1, \ldots, n. \tag{6}$$

Also, as the factors $(\partial - \gamma_i)$ are of order 1 and thus coprime, (6) is a *complete direct decomposition* of $\check{\mathbf{K}}_m[\partial]P$ [4]. For Example 1, a root is $\gamma_1 = a_1 = -t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} + \alpha$. A second root of a fundamental set can be found as a solution of the equation $\gamma_2^{\gamma_2 - \gamma_1} = a_2$: $\gamma_2 = t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} + \alpha$.

A fundamental set of solutions of (3) can be found from a fundamental set of right roots of P $\{\gamma_i, i = 1, \ldots, n\}$ as introduced in Section 2, by solving the *elementary equations*

$$(\partial - \gamma_i)y_i = 0 \Leftrightarrow \frac{dy_i}{dt} y_i^{-1} = \gamma_i, \quad i = 1, \ldots, n. \tag{7}$$

Thus $y_i(t) = e^{\int \gamma_i(t)dt}$, $i = 1, \ldots, n$, which for Example 1 leads to $y_{1,2} = e^{(\pm 2t^{-\frac{1}{2}} + \frac{1}{2}\ln t + \alpha t)}$ which are $\mathbb{C}$-independent. Indeed, $W(y_1, y_2) = \begin{bmatrix} y_1 & y_2 \\ \frac{dy_1}{dt} & \frac{dy_2}{dt} \end{bmatrix}$ and $\det(W(y_1, y_2)) = 2y_1(t)y_2(t)t^{-\frac{3}{2}} \neq 0$ for $t > 0$.

It is thus always possible to find a field extension over which a fundamental set of roots of a given polynomial $P(\partial)$ can be provided. This construction is done without explicit calculation of the solutions of (3):

Proposition 3. *Let $P(\partial) \in \mathbf{K}[\partial]$ and let $\check{\mathbf{K}}_m$ be a Picard–Vessiot extension for (3) as computed above. Then:*

(i) *a fundamental set of roots $\{\gamma_1, \ldots, \gamma_n\}$ of P can be found in $\check{\mathbf{K}}_m$.*
(ii) *the set $\{y_i, i = 1, \ldots, n\}$ where $y_i(t) = e^{\int \gamma_i(t)dt}$ is a fundamental set of solutions of (3).* ◇

5. Conjugacy classes of roots

Consider an autonomous system defined over the Puiseux-type field $\mathbf{K}^\infty = \bigcup_{m \geq 1} \mathbb{C}(t^{1/m})$ and let $\mathbf{R}^\infty = \mathbf{K}^\infty[\partial]$; the above system is, up to a $\mathbf{R}^\infty$-isomorphism, the torsion $\mathbf{R}^\infty$-module T^∞ defined by (3).

Definition 2. Let T^∞ be a torsion $\mathbf{R}^\infty$-module, $T^\infty \cong \mathbf{R}^\infty / \mathbf{R}^\infty P(\partial)$ and let $\{\gamma_1, \ldots, \gamma_n\}$ be a fundamental set of roots of P. The module T^∞ or the autonomous system associated to T^∞ is called *regular near* $+\infty$ if there exists $t_0 \in \mathbb{R}$ such that all functions $\gamma_i : t \to \gamma_i(t)$ are analytic on $(t_0, +\infty)$. ◇

For $T^\infty \cong \mathbf{R}^\infty/\mathbf{R}^\infty P(\partial)$ to be regular near $+\infty$ it is sufficient that the iterative procedure described in Example 1 converges in a finite number of steps. Also, from (7) it is obvious that if an autonomous system is regular near $+\infty$, for any fundamental set of solutions $(y_i)_{i=1,\dots,n}$ there exists $t_0 \in \mathbb{R}$ such that each y_i $(i = 1, \dots n)$ is analytic on $(t_0, +\infty)$.

Extending the ring of scalars to $\check{\mathbf{R}}^\infty = \check{\mathbf{K}}^\infty[\partial]$ where $\check{\mathbf{K}}^\infty = \mathbf{K}^\infty(\gamma_1, \dots, \gamma_n)$ is the field of rational functions of a fundamental set of roots γ_i of P with coefficients in $\mathbf{K}^\infty$, the decomposition of type (6) leads to the following direct sum of modules

$$\check{\mathbf{T}}^\infty \cong \check{\mathbf{R}}^\infty/\check{\mathbf{R}}^\infty P(\partial) \cong \oplus_i \check{\mathbf{R}}^\infty/\check{\mathbf{R}}^\infty(\partial - \gamma_i) \tag{8}$$

which is a complete direct decomposition of $\check{\mathbf{T}}^\infty = \check{\mathbf{R}}^\infty \otimes_{\mathbf{R}^\infty} T^\infty$. For Example 1, one obtains $\check{\mathbf{T}}^\infty \cong \check{\mathbf{R}}^\infty/\check{\mathbf{R}}^\infty(\partial + t^{-\frac{3}{2}} - \frac{1}{2}t^{-1} - \alpha) \oplus \check{\mathbf{R}}^\infty/\check{\mathbf{R}}^\infty(\partial - t^{-\frac{3}{2}} - \frac{1}{2}t^{-1} - \alpha)$.

In the module-based framework, (3) is not the unique differential equation which defines the autonomous system. Indeed, if $\overline{T}^\infty \cong T^\infty$ is a module over $\mathbf{R}^\infty$ defined by the equation

$$\overline{P}(\partial)\overline{y} = 0 \tag{9}$$

then (3) and (9) define the same autonomous system. We say also that $\overline{P}(\partial)$ and $P(\partial)$ are similar [4], written $\overline{P}(\partial) \sim P(\partial)$, when $\mathbf{R}^\infty/\mathbf{R}^\infty\overline{P} \cong \mathbf{R}^\infty/\mathbf{R}^\infty P$. Each γ_i is conjugated with some root $\overline{\gamma}_j$ of $\overline{P}(\partial)$ over $\check{\mathbf{K}}^\infty$. The link between the roots of similar skew polynomials has been studied in [15] where it was shown that there exists $Q \in \mathbf{R}^\infty$ such that $\overline{\gamma}_i = \gamma_i^{Q(\gamma_i)}$ and $V(Q) \cap V(P) = \emptyset$ where for $F \in \mathbf{R}^\infty$, $V(F) = \{x \in \check{\mathbf{K}}^\infty, F(x) = 0\}$. The conjugacy class of any element γ_i of a fundamental set of roots is thus of the form

$$\Delta_{\check{\mathbf{K}}^\infty}(\gamma_i) = \left\{\overline{\gamma}_i = \gamma_i + \frac{\mathrm{d}c}{\mathrm{d}t}c^{-1}, 0 \neq c = Q(\gamma_i), Q \in \mathbf{R}^\infty\right\}. \tag{10}$$

It can be shown that in the situation above where c is the evaluation of a polynomial in a given γ_i, $\frac{\mathrm{d}c}{\mathrm{d}t} \to 0$ as $t \to \infty$. Thus, the conjugacy class (10) is infinite but such that $t \to \infty \Rightarrow \gamma_i - \overline{\gamma}_i \to 0, \forall \overline{\gamma}_i \in \Delta_{\check{\mathbf{K}}^\infty}(\gamma_i)$.

6. Poles and stability of an autonomous LTV system

The stability of a system can be evaluated from the stability of the autonomous system. It can be defined and studied in the intrinsic module framework: consider an autonomous LTV system given by the $\mathbf{R}^\infty$-torsion module T^∞ regular near $+\infty$. Let $(y_i)_{i=1,\dots,n}$ be a fundamental set of solutions of the autonomous system associated with T^∞, $Y = [y_1 \dots y_n]$ and $t_0 \in \mathbb{R}$ such that Y is analytic on $(t_0, +\infty)$. The system is said to be *exponentially stable* if there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\|Y(t)\| \leq \alpha \mathrm{e}^{-\beta(t-t_0)}, \quad t > t_0. \tag{11}$$

It is said *exponentially unstable* if there exist constants $\alpha > 0$ and $\beta > 0$ such that $\|Y(t)\| \geq \alpha \mathrm{e}^{\beta(t-t_0)}, t > t_0$.

Definition 3. Consider an autonomous LTV system Σ given by a torsion $\mathbf{R}^\infty$-module $T^\infty \cong \mathbf{R}^\infty/\mathbf{R}^\infty P(\partial)$ and let $\gamma_i, i = 1, \dots, n$ be a fundamental set of roots of the polynomial $P(\partial)$.

(i) The poles of Σ are the conjugacy classes $\{\Delta_{\check{\mathbf{K}}^\infty}(\gamma_i), i = 1, \dots, n\}$.
(ii) A set $\{\overline{\gamma}_1, \dots, \overline{\gamma}_n\}$ of representatives of these poles (i.e., $\overline{\gamma}_i \in \Delta_{\check{\mathbf{K}}^\infty}(\gamma_i)$) is called *fundamental* if the $\overline{\gamma}_i$'s are P-independent.
(iii) The module of poles of Σ is $\check{\mathbf{T}}^\infty$ defined by (8). ◇

Notice that the poles of Σ only depend on $\check{T}^\infty$ (not on $P(\partial)$). Thus, this definition of poles is intrinsic. For Example 1, a fundamental set of representatives of poles is thus $\gamma_{1,2} = \mp t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} + \alpha$. As a fundamental set of solutions to (3) is given by the elementary equations (7), stability can be directly evaluated in terms of the poles of the system:

Theorem 1. *The regular near $+\infty$ autonomous system Σ defined by the $\mathbf{R}^\infty$-torsion module T^∞ is exponentially stable if and only if*

$$\limsup_{t\to+\infty} \mathrm{Re}\{\gamma_i(t)\} < 0, \quad i = 1, \dots, n \tag{12}$$

where $\{\gamma_1, \dots, \gamma_n\}$ is any fundamental set of representatives of poles of Σ. This system is exponentially unstable if, and only if there exists an index $i \in \{1, \dots, n\}$ and $\overline{\gamma}_i \in \Delta_{\check{\mathbf{K}}^\infty}(\gamma_i)$ such that

$$\liminf_{t\to+\infty} \mathrm{Re}\{\overline{\gamma}_i(t)\} > 0. \quad \diamond \tag{13}$$

Proof. Let $\gamma_1, \dots, \gamma_n$ be a fundamental set of representatives of poles of the autonomous system. A definition equation of T^∞ is (3), where P is the minimal polynomial of $\gamma_1, \dots, \gamma_n$. A fundamental set of solutions of (3) is provided by the solutions y_i of the elementary equations (7). If (12) holds, then $\mathrm{Re}\{\gamma_i(t)\} \leq -\beta_i$ for t large enough, for some $\beta_i > 0, \forall i = 1, \dots, n$. It follows that for $i = 1, \dots, n$ and a time t_1 there exist finite, positive constants α_i and β_i such that

$$|\mathrm{e}^{\int_{t_1}^t \gamma_i d\tau}| \leq \alpha_i \mathrm{e}^{-\beta_i(t-t_1)}, \quad \forall\, t \geq t_1 > t_0 \Leftrightarrow |y_i| \leq \alpha_i \mathrm{e}^{-\beta_i(t-t_1)}, \quad t \geq t_1 > t_0 \tag{14}$$

from which one concludes that any solution $Y(t)$ satisfies (11).

Let now $\overline{\gamma}_1, \dots, \overline{\gamma}_n$ be another set of poles computed for $\check{\overline{T}}^\infty \cong \check{T}^\infty$. Then, as shown in Section 5, $\overline{\gamma}_i$ is conjugated to γ_i by equation $\overline{\gamma}_i = \gamma_i + \frac{\mathrm{d}c}{\mathrm{d}t}c^{-1}$ of (10) and $\gamma_i - \overline{\gamma}_i \to 0$ as $t \to \infty$. It follows that, if $\gamma_1, \dots, \gamma_n$ satisfy condition (14), then the same condition is satisfied by any set $\overline{\gamma}_1, \dots, \overline{\gamma}_n$ of poles from the conjugacy class (10) as $t \to \infty$ and thus the solutions $\overline{y}_i = \mathrm{e}^{\int \overline{\gamma}_i d\tau}$ also satisfy (11). Conversely, if the system is exponentially stable, then for any solution y_i of a fundamental set of solutions of (3) associated to $\check{T}^\infty$ there exist $\alpha_i > 0, \beta_i > 0$ to satisfy (14) where $\gamma_i = \frac{\mathrm{d}y_i}{\mathrm{d}t}y_i^{-1}$ from which (12) follows. ◇

Conditions for asymptotic stability can be obtained in the same manner. Due to the space limitation, their statement is left to the reader. For the system in Example 1, condition (12) is satisfied if and only if $\alpha < 0$ and condition (13) is satisfied if and only if $\alpha > 0$. The system is thus exponentially stable if and only if $\alpha < 0$, exponentially unstable if and only if $\alpha > 0$ and when $\alpha = 0$ the system is neither exponentially stable nor exponentially unstable. It can be shown that in the latter case the system is however unstable. Indeed, the solutions of (3) with $P(\partial)$ of Example 1 are $y(t) = \alpha_1 \mathrm{e}^{\int(-t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} + \alpha)\mathrm{d}t} + \alpha_2 \mathrm{e}^{\int(t^{-\frac{3}{2}} + \frac{1}{2}t^{-1} + \alpha)\mathrm{d}t} = \alpha_1 \mathrm{e}^{(2t^{-\frac{1}{2}} + \frac{1}{2}\ln(t) + \alpha t)} + \alpha_2 \mathrm{e}^{(-2t^{-\frac{1}{2}} + \frac{1}{2}\ln(t) + \alpha t)}, \alpha_1, \alpha_2 \in \mathbb{C}$. Thus, as $t \longrightarrow +\infty$, $|\, y(t)\, | \longrightarrow 0$ exponentially if $\alpha < 0$ and $|\, y(t)\, | \longrightarrow +\infty$ exponentially if $\alpha > 0$. Notice that in case $\alpha = 0$, also $|\, y(t)\, | \longrightarrow +\infty$ but not exponentially.

7. Modules of poles and zeros of an LTV control system

The poles and zeros of the LTI systems have been defined and studied in the module-based framework in [1] where to each kind of pole or zero it was associated a module deduced from the one which defines the system. This approach has been used in [2] to define the structure *at infinity* of an LTV system in an

intrinsic way. Next, these entities of modules of system poles and zeros have been proposed in [3] for the finite structure of LTV systems also. This framework can now be fully validated by considering the field extensions and constructions described above which allows one to define the poles of a given LTV system in the general case where they don't belong to the initial field of definition of the system $\mathbf{K}^\infty$. Indeed, let $M_{sp} \cong \check{M}^\infty/[u]_{\check{\mathbf{R}}^\infty}$, where $\check{M}^\infty = \check{\mathbf{R}}^\infty \otimes_{\mathbf{R}^\infty} M^\infty$ ($\otimes$ denotes the tensor product) be the *module of poles* of the LTV system defined by the $\mathbf{R}^\infty$-module M^∞. Let $B(\partial)$ be a definition matrix of M_{sp} ($B(\partial)w = 0$) and consider its *Jacobson–Teichmüller normal form* [4] over $\check{\mathbf{K}}^\infty$: there exist unimodular matrices U, V with entries in $\check{\mathbf{R}}^\infty$ such that $UBV^{-1} = \mathrm{diag}\{1, \ldots, 1, P_m(\partial)\}$. P_m is defined up to similarity and $P_m \cong P$, where P is the polynomial in (3) of M_{sp}. A set of poles of the LTV system defined by M^∞ is thus a fundamental set of roots of P_m. Let $\partial x = Ax$ be a state-space representation of M_{sp} ($A \in (\check{\mathbf{K}}^\infty)^{n\times n}$). The same kind of characterization is obtained if the similarity of matrices [4] defining the same module is considered: $\partial I - A \sim \mathrm{diag}\{1, \ldots, 1, P(\partial)\}$ and the *right eigenspectrum of A* is $\bigcup_{f\in\Delta(P)} V(f)$ where $\Delta(P) = \{f \mid f \sim P\}$ [15]. The family of the sets of poles of the LTV system defined by M^∞ is thus the family of the right eigenspectrums of the state matrices of M_{sp}.

The same construction can be made for the zeros if one considers a vector z of output variables of an LTV system defined by the $\mathbf{R}^\infty$- module M^∞. Let y be a generator of $\mathcal{T}(M^\infty/[z]_{\mathbf{R}^\infty})$ where $\mathcal{T}(.)$ denotes the torsion submodule. There exists $Z(\partial) \in \mathbf{K}^\infty[\partial]$ such that $Z(\partial)y = 0$ and let $z_1, \ldots, z_l \in \mathbf{K}_l$ be a fundamental set of roots of Z. Let now $\widehat{\mathbf{K}}^\infty = \mathbf{K}^\infty(\gamma_1, \ldots, \gamma_n, z_1, \ldots, z_l)$. The *module of the invariant zeros* of the LTV system defined by M^∞ is $M_{iz} \cong \mathcal{T}(\widehat{M}^\infty/[z]_{\widehat{\mathbf{R}}^\infty})$ where $\widehat{M}^\infty = \widehat{\mathbf{R}}^\infty \otimes_{\mathbf{R}^\infty} M^\infty$, $\widehat{\mathbf{R}}^\infty = \widehat{\mathbf{K}}^\infty[\partial]$. Torsion modules can be introduced in a similar way for the transmission poles/zeros and the decoupling zeros. For this, the following result is used:

Lemma 1. *Let M be a $\mathbf{R}$-module and $N \subseteq M$ be a submodule of M. Consider $F \subseteq N$ a free submodule of N such that $N = \mathcal{T}(N) \oplus F$. Then $\mathcal{T}(N) \subseteq \mathcal{T}(M)$ and there exists $\Phi_N \subseteq M$ a free submodule of M such that $M = \mathcal{T}(M) \oplus \Phi_N$ and $F \subseteq \Phi_N$.* ◇

In the definitions below, $N = [y, u]_{\widehat{\mathbf{R}}^\infty}$ and Φ_N is denoted by $\widehat{\Phi}^\infty$. Modules of finite poles and zeros of LTI systems have already been defined and studied in [1]; the same remain valid in the LTV case:

- $M_{idz} = \mathcal{T}(\widehat{M}^\infty)$ is the *module of the input-decoupling zeros*.
- $M_{odz} = \widehat{M}^\infty/[y, u]_{\widehat{\mathbf{R}}^\infty}$ is the *module of output-decoupling zeros*.
- $M_{iodz} = \mathcal{T}(\widehat{M}^\infty)/(\mathcal{T}(\widehat{M}^\infty) \cap [y, u]_{\widehat{\mathbf{R}}^\infty})$ is the *module of input–output decoupling zeros*.
- $M_{hm} = \widehat{M}^\infty/(\widehat{\Phi}^\infty \cap [y, u]_{\widehat{\mathbf{R}}})$ is the *module of hidden modes*.
- $M_{tz} = (\widehat{\Phi}^\infty \cap [y, u]_{\widehat{\mathbf{R}}^\infty})/(\widehat{\Phi}^\infty \cap [y]_{\widehat{\mathbf{R}}^\infty})$ is the *module of transmission zeros*.
- $M_{tp} = (\widehat{\Phi}^\infty \cap [y, u]_{\widehat{\mathbf{R}}^\infty})/[u]_{\widehat{\mathbf{R}}^\infty}$ is the *module of transmission poles*.
- $M_{cp} = \widehat{\Phi}^\infty/[u]_{\widehat{\mathbf{R}}^\infty}$ is the *module of controllable poles*.
- $M_{op} = [y, u]_{\widehat{\mathbf{R}}^\infty}/[u]_{\widehat{\mathbf{R}}^\infty}$ is the *module of observable poles*.

Case of input-decoupling zeros and controllability: Let Σ be an LTV control system with input $u = (u_i)_{i=1,\ldots,m}$, associated with an $\mathbf{R}^\infty$-module M^∞; the module $[u]_{\mathbf{R}^\infty}$ is free of rank m [6]. A state-space representation can be found for the uncontrollable part of Σ, associated with the torsion module $\mathcal{T}(M)$ [6]. This state-space representation is of the form $\dot{x}_{\bar{c}} = A_{\bar{c}}x_{\bar{c}}$ and the input-decoupling zeros are given by the right eigenspectrum of $A_{\bar{c}}$ (over $\widehat{\mathbf{K}}^\infty$). Here $x_{\bar{c}}$ is a column whose entries are a basis of the finite $\mathbf{K}^\infty$-vector space $\mathcal{T}(M)$. We have the decomposition (see [4]) $M = \mathcal{T}(M) \oplus \Phi$ where Φ is a free module. Next, using Lemma 1 with $N = [u]_{\mathbf{R}^\infty}$ and $\Phi_N = \Phi$, $[u]_{\mathbf{R}^\infty} \subseteq \Phi$ and $M/\mathcal{T}(M) \cong \Phi$ is defined by a state-space equation of the form $\dot{\bar{x}}_c = A_c\bar{x}_c + B_c\bar{u}$ where the entries of $\bar{u}$ are the canonical images of those of u in $M/\mathcal{T}(M)$. This state-space system is controllable, which means that the matrix $[\partial I - A_c \;\; B_c]$ is right-invertible over $\mathbf{R}^\infty$. This is equivalent to the behavioral controllability of this system [19]. Finally, Σ can be represented by a state-space form $\dot{x}_c = A_cx_c + A_{c\bar{c}}x_{\bar{c}} + B_cu$, $\dot{x}_{\bar{c}} = A_{\bar{c}}x_{\bar{c}}$ which is the classical Kalman decomposition [11] with respect to controllability/uncontrollability. The general Kalman decomposition can be obtained in the same way, with a similar interpretation.

Case of transmission zeros: They have the usual significance, i.e., they give the class of nonzero inputs for which a zero output is obtained as shown on the following example:

Example 2. Let $G(\partial) = \begin{bmatrix} 1 & (\partial+t)^{-1} & (\partial+1)^{-1}(\partial+2t) \\ 0 & 0 & (\partial+1)^{-1} \end{bmatrix}$. A factorization of G is provided by computing the left common multiple of $(\partial + 1)$ and $(\partial + t)$: $P(\partial) = [\partial + 1, \partial + t]_l = \partial^2 + \frac{2-t^2}{1-t}\partial + \frac{t^2-t-1}{t-1}$. Then, $G(s) = D_l^{-1}N_l$, where $D_l = \begin{bmatrix} P(\partial) & 0 \\ 0 & \partial+1 \end{bmatrix}$, $N_l = \begin{bmatrix} P(\partial) & \partial - \frac{t-2}{1-t} & \left(\partial - \frac{-t+t^2-1}{1-t}\right)(\partial+2t) \\ 0 & 0 & 1 \end{bmatrix}$. Notice that D_l, N_l are also left-coprime (when it is not the case, such a pair must be constructed from D_l, N_l, *i.e.*, a greatest left factor must be eliminated) and thus they are matrices of definition of the system modules of poles and respectively transmission zeros. It follows that a fundamental set of representatives of the transmission poles is $\{-1, -t\}$ and one for the transmission zeros is $\{\frac{t-2}{1-t}\}$. The latter is unique since it consists of only one element. Therefore, there exist a nonzero $\widehat{\mathbf{R}}^\infty$-linear combination $\bar{u}$ of the inputs satisfying $(\partial - \frac{t-2}{1-t})\bar{u} = 0$ and an $\widehat{\mathbf{R}}^\infty$-linear combination $\bar{y}$ of the outputs, with nonzero coefficients, such that $\bar{y} = 0$ for the above inputs. ◇

The same relations as in the LTI case [1] exist between the poles and zeros defined above. They are consequences of inclusion relations of the torsion modules to which they are associated. For any torsion $\widehat{\mathbf{R}}^\infty$-module T of poles or zeros defined above, let $\Gamma(T) = \cup_i\{\Delta_{\widehat{\mathbf{K}}^\infty}(\gamma_1), \ldots, \Delta_{\widehat{\mathbf{K}}^\infty}(\gamma_n)\}$. The following result is used:

Proposition 4. *Let $T_2 \subseteq T_1$ be two finitely generated torsion modules over $\widehat{\mathbf{R}}^\infty$. Then $\Gamma(T_1) = \Gamma(T_2)\dot{\cup}\Gamma(T_1/T_2)$ where $\dot{\cup}$ denotes the disjoint union.* ◇

This leads to the usual relations between the sets of poles and zeros:

- $\Gamma(M_{hm}) = \Gamma(M_{idz})\dot{\bigcup}\Gamma(M_{odz}) \setminus \Gamma(M_{iodz})$
- $\Gamma(M_{sp}) = \Gamma(M_{idz})\dot{\bigcup}\Gamma(M_{cp}) = \Gamma(M_{odz})\dot{\bigcup}\Gamma(M_{op}) = \Gamma(M_{tp})\dot{\bigcup}\Gamma(M_{hm})$
- $\Gamma(M_{tz})\dot{\bigcup}\Gamma(M_{iodz}) \subseteq \Gamma(M_{iz})$.

For example, $M_{odz} \subseteq M_{sp}$ and $M_{odz} \cong M_{sp}/M_{op}$. Using Proposition 4 with $T_1 = M_{sp}$ and $T_2 = M_{odz}$ one gets $\Gamma(M_{sp}) = \Gamma(M_{odz})\dot{\bigcup}\Gamma(M_{op})$.

8. A physical example

A bandwidth adaptation of linear filters is sometimes needed to fulfil the specifications of several classes of industrial applications. Such an example is treated in [21] where a time-varying bandwidth filter of the form

$$P(\partial)u_{out} = u_{in},$$
$$P(\partial) = \partial^2 + \left[2\xi\omega_n(t) - \frac{d\omega_n}{dt}(\omega_n(t))^{-1}\right]\partial + (\omega_n(t))^2 \quad . \tag{15}$$

is used in the design of a missile autopilot. In order to cope with the requirements on the tracking performance and on the actuator rate limit, the control is filtered by a block of type (15) which is supposed to reduce the acceleration and the rate of the control in case of tracking of abrupt trajectories and, on the contrary, to have little influence on smooth trajectories which can be tracked without limiting the actuator. These two requirements cannot be achieved with a fixed-parameter filter and a well-chosen time dependance $t \mapsto \omega_n(t)$ in (15) of the bandwidth should be considered. Consider the autonomous part

$$P(\partial)y = 0 \tag{16}$$

of (15). For $0 < \xi < 1$, the solutions of (16) can be easily found to be $y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$, $y_{1,2}(t) = \mathrm{e}^{(-\xi \pm \mathrm{i}\sqrt{1-\xi^2}) \int \omega_n(t)\mathrm{d}t}$ where $\mathrm{i}^2 = -1$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. According to the elementary equations (7), to the fundamental set of solutions $\{y_1(t), y_2(t)\}$ above corresponds the fundamental set of roots $\{\gamma_1(t), \gamma_2(t)\}$ of P: $\gamma_{1,2} = -\xi\omega_n(t) \pm \mathrm{i}\omega_n(t)\sqrt{1-\xi^2}$. As the bandwidth variation is necessarily such that $\omega_n(t) \geq \omega_0 > 0$, $\forall t$, condition (12) of Theorem 1 is satisfied and the filter (15) is thus exponentially stable.

9. Extensions to the nonlinear case

A nonlinear system Σ over a differential ground field $\mathbf{k}$ is a finitely generated extension $\mathbf{L}/\mathbf{k}$. Denoting by $d_{\mathbf{L}/\mathbf{k}}$ the Kähler differential [10], the $\mathbf{L}[\partial]$-module $\Omega_{\mathbf{L}/\mathbf{k}}$ spanned by the elements $d_{\mathbf{L}/\mathbf{k}}a$, $a \in \mathbf{L}$ is the tangent linear system Σ_T associated with Σ [8]. Σ_T is an LTV system; its transfer matrix $G(\partial)$ can be calculated as in [7]. The above theory can be applied to Σ_T with $\mathbf{k} = \widehat{\mathbf{K}}^{\infty}$, and Σ is said to be exponentially stable if so is Σ_T (in accordance with the Lyapunov theory). $G(\partial)$ can be called the transfer matrix of Σ (as well as of Σ_T) and has connections with that calculated in [9].

10. Conclusion

Poles and zeros has been defined for LTV systems by extending the field over which the system is given. It has been shown that these definitions are consistent; as a matter of fact, the system poles are connected to the stability of the system and the invariant zeros to the zeroing of the output for non zero inputs, when the field extension is a Picard–Vessiot one for the system. Poles and zeros are not unique, as opposed to their conjugacy classes; this is consistent with the intrinsic module-based approach for LTV systems used in this paper.

References

[1] H. Bourles, M. Fliess, Poles and zeros of linear systems: An intrinsic approach, Int. J. Control 68 (1997) 897–922.
[2] H. Bourles, B. Marinescu, Poles and zeros at infinity of linear time-varying systems, IEEE Trans. Automat. Control 44 (1999) 1981–1985.
[3] H. Bourles, Structural Properties of Discrete and Continuous Linear Time-Varying Systems: A Unified Approach, in: F. Lamnabhi-Lagarrigue, A. Loría, E. Panteley (Eds.), Advanced Topics in Control Systems Theory, Springer, 2005, pp. 225–280 (Chapter 6).
[4] P.M. Cohn, Free Rings and Their Relations, Academic Press, 1985.
[5] J. Delenclos, A. Leroy, Noncommutative symmetric functions and w-polynomials, arXiv e-print, http://arxiv.org/abs/math.RA/0606614.
[6] M. Fliess, Some basic structural properties of generalized linear systems, Syst. Control Lett. 15 (1990) 391–396.
[7] M. Fliess, Une interpétation algebrique de la transformée de Laplace et des matrices de transfert, Linear Algebra and its Applications 203 (1990) 429–443.
[8] M. Fliess, S.T. Glad, An Algebraic Approach to Linear and Nonlinear Control, in: H.L. Trentelman, J.C. Willems (Eds.), Essays on Control: Perspectives in the Theory and its Applications, Birkhäuser, 1993, pp. 223–267.
[9] M. Halas, An algebraic framework generalizing the concept of transfer functions to nonlinear systems, Automatica 44 (5) (2008) 1181–1190.
[10] J. Johnson, Kähler differentials and differential algebra, Ann. of Math., 2nd Ser. 89 (1) (1969) 92–98.
[11] R.E. Kalman, Mathematical description of linear dynamical systems, J. SIAM Control, Ser. A 1 (2) (1963) 152–192.
[12] E.W. Kamen, The poles and zeros of a linear time-varying system, Linear Algebra Appl. 98 (1998) 263–289.
[13] T.Y. Lam, A. Leroy, Vandermonde and Wronskian matrices over division rings, J. Algebra 119 (1988) 306–336.
[14] T.Y Lam, A. Leroy, Wedderburn polynomials over division rings, I, J. Pure Appl. Algebra 186 (2004) 43–76.
[15] T.Y Lam, A. Leroy, Ozturk, Wedderburn polynomials over division rings, II, 2007, http://arxiv.org/abs/0706.3515.
[16] B. Malgrange, Systèmes différentiels à coefficients constants, Séminaire Bourbaki 246 (1962–1963) 1–11.
[17] R.T. O'Brien, P.A. Iglesias, On the poles and zeros of linear, time-varying systems, IEEE Trans. Circuit Syst.-I: Fund. Theory and Applications 48 (5) (2001) 565–577.
[18] P. van der Put, M.F. Singer, Galois Theory of Linear Differential Equations, Springer, 2003.
[19] E. Zerz, An algebraic analysis approach to linear-time varying systems, IMA J. Math. Contr. Inform. 23 (2006) 113–126.
[20] J.J. Zhu, Series and parallel d-spectra for multi-input-multi-output linear time-varying systems, in: Proc. of the South-Eastern Symposium on Systems Theory, Baton Rouge, LA-US, 1995, pp. 125–129.
[21] J.J. Zhu, M.C. Mickle, Missile autopilot design using a new linear time-varying control technique, J. Guid. Control Dynam. 20 (1) (1997) 150–157.